FADDEEV, SOMINSKY | PROBLEMS IN HIGHER ALGEBRA

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ИЗДАТЕЛЬСТВО МИР

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СБОРНИК ЗАДАЧ ПО ВЫСШЕЙ АЛГЕБРЕ

ИЗДАТЕЛЬСТВО «НАУКА» МОСКВА

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PROBLEMS IN HIGHER ALGEBRA

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TO THE READER

Mir Publishers would be grateful for your comments on the content, translation and design of this book. We would also be pleased to receive any other suggestions you may wish to make.

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INTRODUCTION

This book of problems in higher algebra grew out of a course of instruction at the Leningrad State University and the Herzen Pedagogical Institute. It is designed for students of universities and teacher's colleges as a problem book in higher algebra.

The problems included here are of two radically different types. On the one hand, there are a large number of numerical examples aimed at developing computational skills and illustrating the basic propositions of the theory. The authors believe that the number of problems is sufficient to cover work in class, at home and for tests.

On the other hand, there are a rather large number of problems of medium difficulty and many which will demand all the initiative and ingenuity of the student. Many of the problems of this category are accompanied by hints and suggestions to be found in Part II. These problems are starred.

Answers are given to all problems, some of the problems are supplied with detailed solutions.

The authors

PART I. PROBLEMS

CHAPTER 1
COMPLEX NUMBERS

Sec. 1. Operations on Complex Numbers

1. $(1+2i)x+(3-5i)y=1-3i.$
Find x and y, taking them to be real.

2. Solve the following system of equations; x, y, z, t are real:

$$(1+i)x+(1+2i)y+(1+3i)z+(1+4i)t=1+5i,$$

$$(3-i)x+(4-2i)y+(1+i)z+4it=2-i.$$

3. Evaluate i^n, where n is an integer.

4. Verify the identity

$$x^4+4=(x-1-i)(x-1+i)(x+1+i)(x+1-i).$$

5. Evaluate:

(a) $(1+2i)^6$, (b) $(2+i)^7+(2-i)^7$, (c) $(1+2i)^5-(1-2i)^5$.

6. Determine under what conditions the product of two complex numbers is a pure imaginary.

7. Perform the indicated operations:

(a) $\dfrac{1+i\tan\alpha}{1-i\tan\alpha}$, (b) $\dfrac{a+bi}{a-bi}$, (c) $\dfrac{(1+2i)^2-(1-i)^3}{(3+2i)^3-(2+i)^2}$,

(d) $\dfrac{(1-i)^5-1}{(1+i)^5+1}$, (e) $\dfrac{(1+i)^9}{(1-i)^7}$.

8. Evaluate $\dfrac{(1+i)^n}{(1-i)^{n-2}}$, where n is a positive integer.

9. Solve the following systems of equations:

(a) $(3-i)x+(4+2i)y=2+6i$, $(4+2i)x-(2+3i)y=5+4i$;

(b) $(2+i)x+(2-i)y=6$, $(3+2i)x+(3-2i)y=8$;

(c) $x+yi-2z=10$, $x-y+2iz=20$, $ix+3iy-(1+i)z=30$.

10. Evaluate:

(a) $\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^2$,

(b) $\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^3$.

***11.** Let $\omega=-\frac{1}{2}+\frac{i\sqrt{3}}{2}$. Evaluate:

(a) $(a+b\omega+c\omega^2)\,(a+b\omega^2+c\omega)$,

(b) $(a+b)\,(a+b\omega)\,(a+b\omega^2)$,

(c) $(a+b\omega+c\omega^2)^3+(a+b\omega^2+c\omega)^3$,

(d) $(a\omega^2+b\omega)\,(b\omega^2+a\omega)$.

12. Find the conjugates of:
(a) a square, (b) a cube.

***13.** Prove the following theorem:
If as a result of a finite number of rational operations (i. e., addition, subtraction, etc.) on the numbers x_1, $x_2, \ldots, x_n$, we get the number u, then the same operations on the conjugates $\bar{x}_1$, $\bar{x}_2, \ldots, \bar{x}_n$ yield the number $\bar{u}$, which is conjugate to u.

14. Prove that $x^2+y^2=(s^2+t^2)^n$ if $x+yi=(s+ti)^n$.

15. Evaluate:

(a) $\sqrt{2i}$, (b) $\sqrt{-8i}$, (c) $\sqrt{3-4i}$, (d) $\sqrt{-15+8i}$,

(e) $\sqrt{-3-4i}$, (f) $\sqrt{-11+60i}$, (g) $\sqrt{-8+6i}$,

(h) $\sqrt{-8-6i}$, (i) $\sqrt{8-6i}$, (j) $\sqrt{8+6i}$, (k) $\sqrt{2-3i}$,

(l) $\sqrt{4+i}+\sqrt{4-i}$, (m) $\sqrt{1-i\sqrt{3}}$; (n) $\sqrt[4]{-1}$,

(o) $\sqrt[4]{2-i\sqrt{12}}$.

16. $\sqrt{a+bi}=\pm(\alpha+\beta i)$. Find $\sqrt{-a-bi}$.

17. Solve the following equations:

(a) $x^2-(2+i)x+(-1+7i)=0$,

(b) $x^2-(3-2i)x+(5-5i)=0$,

(c) $(2+i)x^2-(5-i)x+(2-2i)=0$.

***18.** Solve the equations and factor the left-hand members into factors with real coefficients:

(a) $x^4+6x^3+9x^2+100=0$,

(b) $x^4+2x^2-24x+72=0$.

19. Solve the equations:

(a) $x^4-3x^2+4=0$, (b) $x^4-30x^2+289=0$.

20. Develop a formula for solving the biquadratic equation $x^4+px^2+q=0$ with real coefficients that is convenient for the case when $\frac{p^2}{4}-q<0$.

Sec. 2. Complex Numbers in Trigonometric Form

21. Construct points depicting the following complex numbers:

$1,\ -1,\ -\sqrt{2},\ i,\ -i,\ i\sqrt{2},\ -1+i,\ 2-3i.$

22. Represent the following numbers in trigonometric form:

(a) 1, (b) -1, (c) i, (d) $-i$, (e) $1+i$,

(f) $-1+i$, (g) $-1-i$, (h) $1-i$, (i) $1+i\sqrt{3}$,

(j) $-1+i\sqrt{3}$, (k) $-1-i\sqrt{3}$, (l) $1-i\sqrt{3}$, (m) $2i$,

(n) -3, (o) $\sqrt{3}-i$, (p) $2+\sqrt{3}+i$.

23. Use tables to represent the following numbers in trigonometric form:

(a) $3+i$, (b) $4-i$, (c) $-2+i$, (d) $-1-2i$.

24. Find the loci of points depicting the complex numbers whose:

(a) modulus is 1, (b) argument is $\frac{\pi}{6}$.

25. Find the loci of points depicting the numbers z that satisfy the inequalities:

(a) $|z|<2$, (b) $|z-i|\leqslant 1$, (c) $|z-1-i|<1$.

26. Solve the equations:

(a) $|x|-x=1+2i$, (b) $|x|+x=2+i$.

***27.** Prove the identity

$|x+y|^2+|x-y|^2=2(|x|^2+|y|^2)$.

What geometrical meaning does it have?

***28.** Prove that any complex number z different from -1, whose modulus is 1, can be represented in the form $z=\frac{1+ti}{1-ti}$ where t is real.

29. Under what conditions is the modulus of the sum of two complex numbers equal to the difference of the moduli of the summands?

30. Under what conditions is the modulus of the sum of two complex numbers equal to the sum of the moduli of the summands?

***31.** z and z' are two complex numbers, $u=\sqrt{zz'}$. Prove that

$$|z|+|z'|=\left|\frac{z+z'}{2}-u\right|+\left|\frac{z+z'}{2}+u\right|.$$

32. Demonstrate that if $|z|<\frac{1}{2}$, then

$$|(1+i)z^3+iz|<\frac{3}{4}.$$

33. Prove that

$$(1+i\sqrt{3})(1+i)(\cos\varphi+i\sin\varphi)=$$
$$=2\sqrt{2}\left[\cos\left(\frac{7\pi}{12}+\varphi\right)+i\sin\left(\frac{7\pi}{12}+\varphi\right)\right].$$

34. Simplify $\dfrac{\cos\varphi+i\sin\varphi}{\cos\psi-i\sin\psi}$.

35. Evaluate $\dfrac{(1-i\sqrt{3})(\cos\varphi+i\sin\varphi)}{2(1-i)(\cos\varphi-i\sin\varphi)}$.

36. Evaluate:

(a) $(1+i)^{25}$, (b) $\left(\dfrac{1+i\sqrt{3}}{1-i}\right)^{20}$,

(c) $\left(1-\dfrac{\sqrt{3}-i}{2}\right)^{24}$, (d) $\dfrac{(-1+i\sqrt{3})^{15}}{(1-i)^{20}}+\dfrac{(-1-i\sqrt{3})^{15}}{(1+i)^{20}}$,

***37.** Prove that

(a) $(1+i)^n=2^{\frac{n}{2}}\left(\cos\dfrac{n\pi}{4}+i\sin\dfrac{n\pi}{4}\right)$,

(b) $(\sqrt{3}-i)^n=2^n\left(\cos\dfrac{n\pi}{6}-i\sin\dfrac{n\pi}{6}\right)$,

n an integer.

*38. Simplify $(1+\omega)^n$, where $\omega=\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}$.

39. Assuming $\omega_1=-\frac{1}{2}+i\frac{\sqrt{3}}{2}$, $\omega_2=-\frac{1}{2}-i\frac{\sqrt{3}}{2}$, determine $\omega_1^n+\omega_2^n$, where n is an integer.

*40. Evaluate $(1+\cos\alpha+i\sin\alpha)^n$.

*41. Prove that if $z+\frac{1}{z}=2\cos\Theta$, then

$$z^m+\frac{1}{z^m}=2\cos m\Theta.$$

42. Prove that $\left(\frac{1+i\tan\alpha}{1-i\tan\alpha}\right)^n=\frac{1+i\tan n\alpha}{1-i\tan n\alpha}$.

43. Extract the roots:

(a) $\sqrt[3]{i}$, (b) $\sqrt[3]{2-2i}$, (c) $\sqrt[4]{-4}$, (d) $\sqrt[6]{1}$, (e) $\sqrt[6]{-27}$.

44. Use tables to extract the following roots:

(a) $\sqrt[3]{2+i}$, (b) $\sqrt[3]{3-i}$, (c) $\sqrt[5]{2+3i}$.

45. Compute:

(a) $\sqrt[6]{\frac{1-i}{\sqrt{3}+i}}$, (b) $\sqrt[8]{\frac{1+i}{\sqrt{3}-i}}$, (c) $\sqrt[6]{\frac{1-i}{1+i\sqrt{3}}}$.

46. Write all the values of $\sqrt[n]{\alpha}$ if you know that β is one of the values.

47. Express the following in terms of $\cos x$ and $\sin x$:

(a) $\cos 5x$, (b) $\cos 8x$, (c) $\sin 6x$, (d) $\sin 7x$.

48. Express $\tan 6\varphi$ in terms of $\tan\varphi$.

49. Develop formulas expressing $\cos nx$ and $\sin nx$ in terms of $\cos x$ and $\sin x$.

50. Represent the following in the form of a first-degree polynomial in the trigonometric functions of angles that are multiples of x:

(a) $\sin^3 x$, (b) $\sin^4 x$, (c) $\cos^5 x$, (d) $\cos^6 x$.

***51.** Prove that

(a) $$2^{2m}\cos^{2m}x = 2\sum_{k=0}^{m-1} C_{2m}^{k}\cos 2(m-k)x + C_{2m}^{m},$$

(b) $$2^{2m}\cos^{2m+1}x = \sum_{k=0}^{m} C_{2m+1}^{k}\cos(2m-2k+1)x,$$

(c) $$2^{2m}\sin^{2m}x = 2\sum_{k=0}^{m-1}(-1)^{m+k} C_{2m}^{k}\cos 2(m-k)x + C_{2m}^{m},$$

(d) $$2^{2m}\sin^{2m+1}x = \sum_{k=0}^{m}(-1)^{m+k} C_{2m+1}^{k}\sin(2m-2k+1)x.$$

***52.** Prove that $2\cos mx = (2\cos x)^m$

$$-\frac{m}{1}(2\cos x)^{m-2} + \frac{m(m-3)}{1\cdot 2}(2\cos x)^{m-4}$$

$$-\ldots + (-1)^p\,\frac{m(m-p-1)(m-p-2)\ldots(m-2p+1)}{p!}$$

$$\times(2\cos x)^{m-2p} + \ldots$$

***53.** Express $\dfrac{\sin mx}{\sin x}$ in terms of $\cos x$.

***54.** Find the sums:

(a) $1 - C_n^2 + C_n^4 - C_n^6 + \ldots,$

(b) $C_n^1 - C_n^3 + C_n^5 - C_n^7 + \ldots$

***55.** Prove that

(a) $$1 + C_n^4 + C_n^8 + \ldots = \frac{1}{2}\left(2^{n-1} + 2^{\frac{n}{2}}\cos\frac{n\pi}{4}\right),$$

(b) $$C_n^1 + C_n^5 + C_n^9 + \ldots = \frac{1}{2}\left(2^{n-1} + 2^{\frac{n}{2}}\sin\frac{n\pi}{4}\right),$$

(c) $$C_n^2 + C_n^6 + C_n^{10} + \ldots = \frac{1}{2}\left(2^{n-1} - 2^{\frac{n}{2}}\cos\frac{n\pi}{4}\right),$$

(d) $$C_n^3 + C_n^7 + C_n^{11} + \ldots = \frac{1}{2}\left(2^{n-1} - 2^{\frac{n}{2}}\sin\frac{n\pi}{4}\right).$$

***56.** Find the sum

$$C_n^1 - \frac{1}{3}C_n^3 + \frac{1}{9}C_n^5 - \frac{1}{27}C_n^7 + \dots$$

57. Prove that $(x+a)^m + (x+a\omega)^m + (x+a\omega^2)^m = 3x^m + 3C_m^3 x^{m-3}a^3 + \dots + 3C_m^n x^{m-n}a^n$, where $\omega = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$ and n is the largest integral multiple of 3 not exceeding m.

58. Prove that

(a) $1 + C_n^3 + C_n^6 + \dots = \frac{1}{3}\left(2^n + 2\cos\frac{n\pi}{3}\right)$,

(b) $C_n^1 + C_n^4 + C_n^7 + \dots = \frac{1}{3}\left(2^n + 2\cos\frac{(n-2)\pi}{3}\right)$,

(c) $C_n^2 + C_n^5 + C_n^8 + \dots = \frac{1}{3}\left(2^n + 2\cos\frac{(n-4)\pi}{3}\right)$.

59. Compute the sums:

(a) $1 + a\cos\varphi + a^2\cos 2\varphi + \dots + a^k\cos k\varphi$,

(b) $\sin\varphi + a\sin(\varphi+h) + a^2\sin(\varphi+2h) + \dots + a^k\sin(\varphi+kh)$,

(c) $\frac{1}{2} + \cos x + \cos 2x + \dots + \cos nx$.

60. Demonstrate that

$$\sin x + \sin 2x + \dots + \sin nx = \frac{\sin\frac{n+1}{2}x \cdot \sin\frac{nx}{2}}{\sin\frac{x}{2}}.$$

61. Find

$$\lim_{n\to\infty}\left(1 + \frac{1}{2}\cos x + \frac{1}{4}\cos 2x + \dots + \frac{1}{2^n}\cos nx\right).$$

62. Prove that if n is a positive integer and Θ is an angle satisfying the condition $\sin\frac{\Theta}{2} = \frac{1}{2n}$, then

$$\cos\frac{\Theta}{2} + \cos\frac{3\Theta}{2} + \dots + \cos\frac{2n-1}{2}\Theta = n\sin n\Theta.$$

63. Show that

(a) $\cos\frac{\pi}{11} + \cos\frac{3\pi}{11} + \cos\frac{5\pi}{11} + \cos\frac{7\pi}{11} + \cos\frac{9\pi}{11} = \frac{1}{2}$,

(b) $\cos\frac{2\pi}{11} + \cos\frac{4\pi}{11} + \cos\frac{6\pi}{11} + \cos\frac{8\pi}{11} + \cos\frac{10\pi}{11} = -\frac{1}{2}$,

(c) $\cos\frac{\pi}{13} + \cos\frac{3\pi}{13} + \cos\frac{5\pi}{13} + \cos\frac{7\pi}{13} + \cos\frac{9\pi}{13} + \cos\frac{11\pi}{13} = \frac{1}{2}$.

64. Find the sums

(a) $\cos a-\cos(a+h)+\cos(a+2h)-\ldots+(-1)^{n-1}\cos[a+(n-1)h]$,

(b) $\sin a-\sin(a+h)+\sin(a+2h)-\ldots+(-1)^{n-1}\sin[a+(n-1)h]$.

65. Prove that if x is less than unity in absolute value, then the series

(a) $\cos\alpha+x\cos(\alpha+\beta)+x^2\cos(\alpha+2\beta)+\ldots+x^n\cos(\alpha+n\beta)+\ldots$,

(b) $\sin\alpha+x\sin(\alpha+\beta)+x^2\sin(\alpha+2\beta)+\ldots+x^n\sin(\alpha+n\beta)+\ldots$

converge and the sums are respectively equal to

$$\frac{\cos\alpha-x\cos(\alpha-\beta)}{1-2x\cos\beta+x^2}, \quad \frac{\sin\alpha-x\sin(\alpha-\beta)}{1-2x\cos\beta+x^2}.$$

66. Find the sums of:

(a) $\cos x+C_n^1\cos 2x+\ldots+C_n^n\cos(n+1)x$,

(b) $\sin x+C_n^1\sin 2x+\ldots+C_n^n\sin(n+1)x$.

67. Find the sums of:

(a) $\cos x-C_n^1\cos 2x+C_n^2\cos 3x-\ldots+(-1)^n C_n^n\cos(n+1)x$,

(b) $\sin x-C_n^1\sin 2x+C_n^2\sin 3x-\ldots+(-1)^n C_n^n\sin(n+1)x$.

***68.** $\overrightarrow{OA_1}$ and $\overrightarrow{OB}$ are vectors depicting 1 and i respectively. From O drop a perpendicular OA_2 on A_1B; from A_2 drop a perpendicular A_2A_3 on OA_1; from A_3, a perpendicular A_3A_4 on A_1A_2, etc. in accordance with the rule: from A_n a perpendicular A_nA_{n+1} is dropped on $A_{n-2}A_{n-1}$. Find the limit of the sum

$$\overrightarrow{OA_1}+\overrightarrow{A_1A_2}+\overrightarrow{A_2A_3}+\ldots$$

***69.** Find the sum

$$\sin^2 x+\sin^2 3x+\ldots+\sin^2(2n-1)x.$$

70. Show that:

(a) $\cos^2 x+\cos^2 2x+\ldots+\cos^2 nx=\dfrac{n}{2}+\dfrac{\cos(n+1)x\sin nx}{2\sin x}$,

(b) $\sin^2 x+\sin^2 2x+\ldots+\sin^2 nx=\dfrac{n}{2}-\dfrac{\cos(n+1)x\sin nx}{2\sin x}$.

***71.** Find the sums of:

(a) $\cos^3 x+\cos^3 2x+\ldots+\cos^3 nx$,

(b) $\sin^3 x+\sin^3 2x+\ldots+\sin^3 nx$.

***72.** Find the sums of:

(a) $\cos x+2\cos 2x+3\cos 3x+\ldots+n\cos nx$,

(b) $\sin x+2\sin 2x+3\sin 3x+\ldots+n\sin nx$.

73. Find $\lim\limits_{n\to\infty}\left(1+\frac{\alpha}{n}\right)^n$ for $\alpha=a+bi$.

74. Definition: $e^\alpha=\lim\limits_{n\to\infty}\left(1+\frac{\alpha}{n}\right)^n$. Prove that

(a) $e^{2\pi i}=1$, (b) $e^{\pi i}=-1$,

(c) $e^{\alpha+\beta}=e^\alpha\cdot e^\beta$, (d) $(e^\alpha)^k=e^{\alpha k}$ for integral k.

Sec. 3. Equations of Third and Fourth Degree

75. Solve the following equations using Cardan's formula:

(a) $x^3-6x+9=0$, (b) $x^3+12x+63=0$,

(c) $x^3+9x^2+18x+28=0$, (d) $x^3+6x^2+30x+25=0$,

(e) $x^3-6x+4=0$, (f) $x^3+6x+2=0$,

(g) $x^3+18x+15=0$, (h) $x^3-3x^2-3x+11=0$,

(i) $x^3+3x^2-6x+4=0$, (j) $x^3+9x-26=0$,

(k) $x^3+24x-56=0$, (l) $x^3+45x-98=0$,

(m) $x^3+3x^2-3x-1=0$, (n) $x^3-6x^2+57x-196=0$,

(o) $x^3+3x-2i=0$, (p) $x^3-6ix+4(1-i)=0$,

(q) $x^3-3abx+a^3+b^3=0$,

(r) $x^3-3abfgx+f^2ga^3+fg^2b^3=0$,

(s) $x^3-4x-1=0$; (t) $x^3-4x+2=0$.

***76.** Using Cardan's formula, prove that

$$(x_1-x_2)^2(x_1-x_3)^2(x_2-x_3)^2=-4p^3-27q^2$$

if x_1, x_2, x_3 are roots of the equation $x^3+px+q=0$.

(The expression $-4p^3-27q^2$ is called the discriminant of the equation $x^3+px+q=0$.)

***77.** Solve the equation

$$(x^3-3qx+p^3-3pq)^2-4(px+q)^3=0.$$

***78.** Derive a formula for solving the equation

$$x^5-5ax^3+5a^2x-2b=0.$$

79. Solve the following equations:

(a) $x^4-2x^3+2x^2+4x-8=0$,

(b) $x^4+2x^3-2x^2+6x-15=0$,

(c) $x^4-x^3-x^2+2x-2=0$,

(d) $x^4-4x^3+3x^2+2x-1=0$,

(e) $x^4-3x^3+x^2+4x-6=0$,

(f) $x^4-6x^3+6x^2+27x-56=0$,

(g) $x^4-2x^3+4x^2-2x+3=0$,

(h) $x^4-x^3-3x^2+5x-10=0$,

(i) $x^4+2x^3+8x^2+2x+7=0$,

(j) $x^4+6x^3+6x^2-8=0$,

(k) $x^4-6x^3+10x^2-2x-3=0$,

(l) $x^4-2x^3+4x^2+2x-5=0$,

(m) $x^4-x^3-3x^2+x+1=0$,

(n) $x^4-x^3-4x^2+4x+1=0$,

(o) $x^4-2x^3+x^2+2x-1=0$,

(p) $x^4-4x^3-2x^2-8x+4=0$,

(q) $x^4-2x^3+3x^2-2x-2=0$,

(r) $x^4-x^3+2x-1=0$,

(s) $4x^4-4x^3+3x^2-2x+1=0$,

(t) $4x^4-4x^3-6x^2+2x+1=0$.

80. Ferrari's method for solving the quartic equation $x^4+ax^3+bx^2+cx+d=0$ consists in representing the left member in the form

$$\left(x^2+\frac{a}{2}x+\frac{\lambda}{2}\right)^2-\left[\left(\frac{a^2}{4}+\lambda-b\right)x^2+\left(\frac{a\lambda}{2}-c\right)x+\left(\frac{\lambda^2}{4}-d\right)\right].$$

Then λ is chosen so that the expression in the square brackets is the square of a first-degree binomial. For this purpose it is necessary and sufficient that

$$\left(\frac{\alpha\lambda}{2}-c\right)^2-4\left(\frac{a^2}{4}+\lambda-b\right)\left(\frac{\lambda^2}{4}-d\right)=0,$$

that is, λ must be a root of some auxiliary cubic equation. Having found λ, factor the left member.

Express the roots of the auxiliary equation in terms of the roots of the fourth-degree equation.

Sec. 4. Roots of Unity

81. Write the following roots of unity of degree

(a) 2, (b) 3, (c) 4, (d) 6, (e) 8, (f) 12, (g) 24.

82. Write the primitive roots of degree

(a) 2, (b) 3, (c) 4, (d) 6, (e) 8, (f) 12, (g) 24.

83. To what exponent do the following belong:

(a) $z_k=\cos\frac{2k\pi}{180}+i\sin\frac{2k\pi}{180}$ for $k=27,\ 99,\ 137$;

(b) $z_k=\cos\frac{2k\pi}{144}+i\sin\frac{2k\pi}{144}$ for $k=10,\ 35,\ 60$?

84. Write out all the 28th roots of unity belonging to the exponent 7.

85. For each of the roots of unity: (a) 16th, (b) 20th, (c) 24th, indicate the exponent it belongs to.

86. Write out the "cyclotomic polynomials" $X_n(x)$ for n equal to:

(a) 1, (b) 2, (c) 3, (d) 4, (e) 5, (f) 6, (g) 7, (h) 8, (i) 9, (j) 10, (k) 11, (l) 12, (m) 15, (n) 105.

***87.** Let ε be a primitive $2n$-th root of unity. Compute the sum $1+\varepsilon+\varepsilon^2+\ldots+\varepsilon^{n-1}$.

***88.** Find the sum of all the nth roots of unity.

***89.** Find the sum of the kth powers of all nth roots of unity.

90. In the expression $(x+a)^m$ substitute in succession, for a, the m mth roots of unity, then add the results.

***91.** Compute $1+2\varepsilon+3\varepsilon^2+\ldots+n\varepsilon^{n-1}$, where ε is an nth root of unity.

***92.** Compute $1+4\varepsilon+9\varepsilon^2+\ldots+n^2\varepsilon^{n-1}$, where ε is an nth root of unity.

93. Find the sums:

(a) $\cos\frac{2\pi}{n}+2\cos\frac{4\pi}{n}+\ldots+(n-1)\cos\frac{2(n-1)\pi}{n}$,

(b) $\sin\frac{2\pi}{n}+2\sin\frac{4\pi}{n}+\ldots+(n-1)\sin\frac{2(n-1)\pi}{n}$.

***94.** Determine the sum of the following primitive roots of unity: (a) 15th, (b) 24th, (c) 30th.

95. Find the fifth roots of unity by solving the equation $x^5-1=0$ algebraically.

96. Using the result of Problem 95, write sin 18° and cos 18°.

***97.** Write the simplest kind of algebraic equation whose root is the length of the side of a regular 14-sided polygon inscribed in a circle of radius unity.

***98.** Decompose x^n-1 into linear and quadratic factors with real coefficients.

***99.** Use the result of Problem 98 to prove the formulas:

(a) $\sin\frac{\pi}{2m}\cdot\sin\frac{2\pi}{2m}\ \ldots\ \sin\frac{(m-1)\pi}{2m}=\frac{\sqrt{m}}{2^{m-1}}$,

(b) $\sin\frac{\pi}{2m+1}\cdot\sin\frac{2\pi}{2m+1}\ \ldots\ \sin\frac{m\pi}{2m+1}=\frac{\sqrt{2m+1}}{2^m}$.

***100.** Prove that $\prod\limits_{k=0}^{n-1}(a+b\varepsilon_k)=a^n+(-1)^{n-1}b^n$

where

$$\varepsilon_k=\cos\frac{2k\pi}{n}+i\sin\frac{2k\pi}{n}.$$

***101.** Prove that

$$\prod_{k=0}^{n-1}(\varepsilon_k^2-2\varepsilon_k\cos\Theta+1)=2(1-\cos n\Theta),$$

if

$$\varepsilon_k=\cos\frac{2k\pi}{n}+i\sin\frac{2k\pi}{n}.$$

102. Prove that

$$\prod_{k=0}^{n-1}\frac{(t+\varepsilon_k)^n-1}{t}=\prod_{k=1}^{n-1}[t^n-(\varepsilon_k-1)^n]$$

where $\varepsilon_k=\cos\frac{2k\pi}{n}+i\sin\frac{2k\pi}{n}$.

***103.** Find all the complex numbers that satisfy the condition $\bar{x}=x^{n-1}$ where $\bar{x}$ is the conjugate of x.

104. Show that the roots of the equation $\lambda(z-a)^n+\mu(z-b)^n=0$, where λ, μ, a, b are complex, lie on one circle, which in a particular case can degenerate into a straight line (n is a natural number).

***105.** Solve the equations:

(a) $(x+1)^m-(x-1)^m=0$, (b) $(x+i)^m-(x-i)^m=0$,

(c) $x^n-nax^{n-1}-C_n^2a^2x^{n-2}-\ldots-a^n=0$.

106. Prove that if A is a complex number with modulus 1, then the equation

$$\left(\frac{1+ix}{1-ix}\right)^m=A$$

has all roots real and distinct.

***107.** Solve the equation

$$\cos\varphi+C_n^1\cos(\varphi+\alpha)x+C_n^2\cos(\varphi+2\alpha)x^2+\ldots+C_n^n\cos(\varphi+n\alpha)x^n=0.$$

Prove the following theorems:

108. The product of an ath root of unity by a bth root of unity is an abth root of unity.

109. If a and b are relatively prime, then x^a-1 and x^b-1 have a unique root in common.

110. If a and b are relatively prime, then all the abth roots of unity are obtained by multiplying the ath roots of unity by the bth roots of unity.

111. If a and b are relatively prime, then the product of a primitive ath root of unity by a primitive bth root of unity is a primitive abth root of unity, and conversely.

112. Denoting by $\varphi(n)$ the number of primitive nth roots of unity, prove that $\varphi(ab)=\varphi(a)\varphi(b)$ if a and b are relatively prime.

***113.** Prove that if $n=p_1^{\alpha_1}p_2^{\alpha_2}\ldots p_k^{\alpha_k}$, where $p_1, p_2, \ldots, p_k$ are distinct primes, then

$$\varphi(n)=n\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\ldots\left(1-\frac{1}{p_k}\right).$$

114. Show that the number of primitive nth roots of unity is even if $n>2$.

115. Write the polynomial $X_p(x)$ where p is prime.

***116.** Write the polynomial $X_{p^m}(x)$ where p is prime.

***117.** Prove that for n odd and greater than unity, $X_{2n}(x) = X_n(-x)$.

118. Prove that if d is made up of prime divisors which enter into n, then each primitive ndth root of unity is a dth root of a primitive nth root of unity, and conversely.

***119.** Prove that if $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$ where $p_1, p_2, \dots, p_k$ are distinct primes, then $X_n(x) = X_{n'}(x^{n''})$ where

$$n' = p_1 p_2 \dots p_k, \quad n'' = \frac{n}{n'}.$$

***120.** Denoting by $\mu(n)$ the sum of the primitive nth roots of unity, prove that $\mu(n) = 0$ if n is divisible by the square of at least one prime number; $\mu(n) = 1$ if n is the product of an even number of distinct prime numbers; $\mu(n) = -1$ if n is the product of an odd number of distinct prime numbers.

121. Prove that $\Sigma \mu(d) = 0$ if d runs through all divisors of the number n, $n \neq 1$.

***122.** Prove that $X_n(x) = \Pi (x^d - 1)^{\mu\left(\frac{n}{d}\right)}$ where d runs through all divisors of n.

***123.** Find $X_n(1)$.

***124.** Find $X_n(-1)$.

***125.** Determine the sum of the products of the primitive nth roots of unity taken two at a time.

***126.** $S = 1 + \varepsilon + \varepsilon^4 + \varepsilon^9 + \dots + \varepsilon^{(n-1)^2}$ where ε is a primitive nth root of unity. Find $|S|$.

CHAPTER 2
EVALUATION OF DETERMINANTS

Sec. 1. Determinants of Second and Third Order

Compute the determinants:

127. (a) $\begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix}$, (b) $\begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix}$, (c) $\begin{vmatrix} \sin\alpha & \cos\alpha \\ -\cos\alpha & \sin\alpha \end{vmatrix}$,

(d) $\begin{vmatrix} a & c+di \\ c-di & b \end{vmatrix}$, (e) $\begin{vmatrix} \alpha+\beta i & \gamma+\delta i \\ \gamma-\delta i & \alpha-\beta i \end{vmatrix}$,

(f) $\begin{vmatrix} \sin\alpha & \cos\alpha \\ \sin\beta & \cos\beta \end{vmatrix}$, (g) $\begin{vmatrix} \cos\alpha & \sin\alpha \\ \sin\beta & \cos\beta \end{vmatrix}$,

(h) $\begin{vmatrix} \tan\alpha & -1 \\ 1 & \tan\alpha \end{vmatrix}$, (i) $\begin{vmatrix} 1+\sqrt{2} & 2-\sqrt{3} \\ 2+\sqrt{3} & 1-\sqrt{2} \end{vmatrix}$,

(j) $\begin{vmatrix} 1 & \log_b a \\ \log_a b & 1 \end{vmatrix}$, (k) $\begin{vmatrix} a+b & b+d \\ a+c & c+d \end{vmatrix}$,

(l) $\begin{vmatrix} a+b & a-b \\ a-b & a+b \end{vmatrix}$, (m) $\begin{vmatrix} x-1 & 1 \\ x^3 & x^2+x+1 \end{vmatrix}$

(n) $\begin{vmatrix} \omega & \omega \\ -1 & \omega \end{vmatrix}$

where $\omega = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$,

(o) $\begin{vmatrix} \varepsilon & 1 \\ -1 & \varepsilon \end{vmatrix}$

where $\varepsilon = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$.

128. (a) $\begin{vmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{vmatrix}$, (b) $\begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}$,

(c) $\begin{vmatrix} a & a & a \\ -a & a & x \\ -a & -a & x \end{vmatrix}$, (d) $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{vmatrix}$,

(e) $\begin{vmatrix} 1 & i & 1+i \\ -i & 1 & 0 \\ 1-i & 0 & 1 \end{vmatrix}$,

(f) $\begin{vmatrix} 1 & \cos\frac{\pi}{3}+i\sin\frac{\pi}{3} & \cos\frac{\pi}{4}+i\sin\frac{\pi}{4} \\ \cos\frac{\pi}{3}-i\sin\frac{\pi}{3} & 1 & \cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3} \\ \cos\frac{\pi}{4}-i\sin\frac{\pi}{4} & \cos\frac{2\pi}{3}-i\sin\frac{2\pi}{3} & 1 \end{vmatrix}$,

(g) $\begin{vmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{vmatrix}$ where $\omega=\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}$,

(h) $\begin{vmatrix} 1 & 1 & \omega \\ 1 & 1 & \omega^2 \\ \omega^2 & \omega & 1 \end{vmatrix}$ where $\omega=\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}$.

Sec. 2. Permutations

129. Write out the transpositions enabling one to go from the permutation 1, 2, 4, 3, 5 to the permutation 2, 5, 3, 4, 1.

130. Assuming that 1, 2, 3, 4, 5, 6, 7, 8, 9 is the initial arrangement, determine the number of inversions in the permutations:

(a) 1, 3, 4, 7, 8, 2, 6, 9, 5; (b) 2, 1, 7, 9, 8, 6, 3, 5, 4;

(c) 9, 8, 7, 6, 5, 4, 3, 2, 1.

131. Assuming 1, 2, 3, 4, 5, 6, 7, 8, 9 to be the initial ordering, choose i and k so that:

(a) the permutation 1, 2, 7, 4, i, 5, 6, k, 9 is even;

(b) the permutation 1, i, 2, 5, k, 4, 8, 9, 7 is odd.

***132.** Determine the number of inversions in the permutation $n, n-1, \ldots, 2, 1$ if the initial permutation is $1, 2, \ldots, n$.

***133.** There are I inversions in the permutation $\alpha_1, \alpha_2, \ldots, \alpha_n$. How many inversions are there in the permutation $\alpha_n, \alpha_{n-1}, \ldots, \alpha_2, \alpha_1$?

134. Determine the number of inversions in the permutations:

(a) $1, 3, 5, 7, \ldots, 2n-1, 2, 4, 6, \ldots, 2n$,

(b) $2, 4, 6, 8, \ldots, 2n, 1, 3, 5, \ldots, 2n-1$

if the initial permutation is $1, 2, \ldots, 2n$.

135. Determine the number of inversions in the permutations:

(a) $3, 6, 9, \ldots, 3n, 1, 4, 7, \ldots, 3n-2, 2, 5, \ldots, 3n-1$,

(b) $1, 4, 7, \ldots, 3n-2, 2, 5, \ldots, 3n-1, 3, 6, \ldots, 3n$

if the initial permutation is $1, 2, 3, \ldots, 3n$.

136. Prove that if $a_1, a_2, \ldots, a_n$ is a permutation with I the number of inversions, then, when returned to its original ordering, the numbers $1, 2, \ldots, n$ form a permutation with the same number of inversions I.

137. Determine the parity of the permutation of the letters *th*, *r*, *m*, *i*, *a*, *g*, *o*, *l* if for the original ordering we take the words (a) logarithm, (b) algorithm.

Compare and explain the results.

Sec. 3. Definition of a Determinant

138. Indicate the signs of the following products that enter into a sixth-order determinant:

(a) $a_{23}a_{31}a_{42}a_{56}a_{14}a_{65}$, (b) $a_{32}a_{43}a_{14}a_{51}a_{66}a_{25}$.

139. Do the following products enter into a 5th-order determinant:

(a) $a_{13}a_{24}a_{23}a_{41}a_{55}$, (b) $a_{21}a_{13}a_{34}a_{55}a_{42}$?

140. Choose i and k so that the product $a_{1i}a_{32}a_{4k}a_{25}a_{53}$ enters into a fifth-order determinant with the plus sign.

141. Write out all the summands that enter into a fourth-order determinant with the plus sign and contain the factor a_{23}.

142. Write out all the summands that enter into a fifth-order determinant and are of the form $a_{14}a_{23}a_{3\alpha_3}a_{4\alpha_4}a_{5\alpha_5}$. What will happen if $a_{14}a_{23}$ is taken outside the parentheses?

143. With what sign does the product of the elements of the principal diagonal enter an nth-order determinant?

144. What sign does the product of elements of the secondary diagonal have in an nth-order determinant?

***145.** Guided solely by the definition of a determinant, prove that the determinant

$$\begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 \\ a_1 & a_2 & 0 & 0 & 0 \\ b_1 & b_2 & 0 & 0 & 0 \\ c_1 & c_2 & 0 & 0 & 0 \end{vmatrix}$$

is zero.

146. Using only the definition of a determinant, evaluate the coefficients of x^4 and x^3 in the expression

$$f(x)=\begin{vmatrix} 2x & x & 1 & 2 \\ 1 & x & 1 & -1 \\ 3 & 2 & x & 1 \\ 1 & 1 & 1 & x \end{vmatrix}.$$

147. Evaluate the determinants:

$$\text{(a)} \begin{vmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 2 & 0 & \dots & 0 \\ 0 & 0 & 3 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & n \end{vmatrix}, \quad \text{(b)} \begin{vmatrix} 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 1 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & 0 & 0 & \dots & 0 & 0 \end{vmatrix},$$

$$\text{(c)} \begin{vmatrix} 1 & a & a & \dots & a \\ 0 & 2 & a & \dots & a \\ 0 & 0 & 3 & \dots & a \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & n \end{vmatrix}.$$

Note: In all problems, determinants are taken to be of order n unless otherwise stated or unless it follows from the conditions of the problem.

148. $F(x)=x(x-1)(x-2)\dots(x-n+1)$.
Compute the determinants:

$$\text{(a)}\quad \begin{vmatrix} F(0) & F(1) & F(2) & & F(n) \\ F(1) & F(2) & F(3) & \dots & F(n+1) \\ \dots & \dots & \dots & \dots & \dots \\ F(n) & F(n+1) & F(n+2) & \dots & F(2n) \end{vmatrix};$$

$$\text{(b)}\quad \begin{vmatrix} F(a) & F'(a) & F''(a) & & F^{(n)}(a) \\ F'(a) & F''(a) & F'''(a) & \dots & F^{(n+1)}(a) \\ \dots & \dots & \dots & \dots & \dots \\ F^{(n)}(a) & F^{(n+1)}(a) & F^{(n+2)}(a) & \dots & F^{(2n)}(a) \end{vmatrix}.$$

Sec. 4. Basic Properties of Determinants

***149.** Prove that an nth-order determinant, each element a_{ik} of which is a complex conjugate of a_{ki}, is equal to a real number.

***150.** Prove that a determinant of odd order is zero if all its elements satisfy the condition

$$a_{ik}+a_{ki}=0$$

(skew-symmetric determinant).

151. The determinant $\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$ is equal to Δ.

To what is the following determinant equal

$$\begin{vmatrix} a_{21} & a_{22} & \dots & a_{2n} \\ a_{31} & a_{32} & \dots & a_{3n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \\ a_{11} & a_{12} & \dots & a_{1n} \end{vmatrix}?$$

152. How is a determinant affected if all columns are written in reversed order?

***153.** What is the sum of

$$\sum \begin{vmatrix} a_{1\alpha_1} & a_{1\alpha_2} & \dots & a_{1\alpha_n} \\ a_{2\alpha_1} & a_{2\alpha_2} & \dots & a_{2\alpha_n} \\ \dots & \dots & \dots & \dots \\ a_{n\alpha_1} & a_{n\alpha_2} & \dots & a_{n\alpha_n} \end{vmatrix}$$

if the summation is taken over all permutations of $\alpha_1, \alpha_2, \dots, \alpha_n$?

***154.** Solve the equations:

$$\text{(a)}\quad \begin{vmatrix} 1 & x & x^2 & \dots & x^{n-1} \\ 1 & a_1 & a_1^2 & \dots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \dots & a_2^{n-1} \\ \dots & \dots & \dots & \dots & \dots \\ 1 & a_{n-1} & a_{n-1}^2 & \dots & a_{n-1}^{n-1} \end{vmatrix} = 0$$

where $a_1, a_2, \dots, a_{n-1}$ are all distinct;

$$\text{(b)}\quad \begin{vmatrix} 1 & 1 & 1 & & 1 \\ 1 & 1-x & 1 & & 1 \\ 1 & 1 & 2-x & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & 1 & \dots & (n-1)-x \end{vmatrix} = 0;$$

$$\text{(c)}\quad \begin{vmatrix} a_1 & a_2 & a_3 & & a_n \\ a_1 & a_1+a_2-x & a_3 & & a_n \\ a_1 & a_2 & a_2+a_3-x & \dots & a_n \\ \dots & \dots & \dots & \dots & \dots \\ a_1 & a_2 & a_3 & \dots & a_{n-1}+a_n-x \end{vmatrix} = 0.$$

***155.** The numbers 204, 527 and 255 are divisible by 17. Prove that 17 divides

$$\begin{vmatrix} 2 & 0 & 4 \\ 5 & 2 & 7 \\ 2 & 5 & 5 \end{vmatrix}.$$

***156.** Compute the determinant

$$\begin{vmatrix} \alpha^2 & (\alpha+1)^2 & (\alpha+2)^2 & (\alpha+3)^2 \\ \beta^2 & (\beta+1)^2 & (\beta+2)^2 & (\beta+3)^2 \\ \gamma^2 & (\gamma+1)^2 & (\gamma+2)^2 & (\gamma+3)^2 \\ \delta^2 & (\delta+1)^2 & (\delta+2)^2 & (\delta+3)^2 \end{vmatrix}.$$

157. Prove that

$$\begin{vmatrix} b+c & c+a & a+b \\ b_1+c_1 & c_1+a_1 & a_1+b_1 \\ b_2+c_2 & c_2+a_2 & a_2+b_2 \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}.$$

158. Simplify the determinant $\begin{vmatrix} am+bp & an+bq \\ cm+dp & cn+dq \end{vmatrix}$ by expanding it into summands.

159. Find the sum of the cofactors of all elements of the determinants:

$$\text{(a)} \begin{vmatrix} a_1 & 0 & 0 & \dots & 0 \\ 0 & a_2 & 0 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & a_n \end{vmatrix}; \quad \text{(b)} \begin{vmatrix} 0 & 0 & \dots & 0 & a_1 \\ 0 & 0 & \dots & a_2 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_n & 0 & \dots & 0 & 0 \end{vmatrix}.$$

160. Expand the following determinant by the elements of the third row and evaluate:

$$\begin{vmatrix} 1 & 0 & -1 & -1 \\ 0 & -1 & -1 & 1 \\ a & b & c & d \\ -1 & -1 & 1 & 0 \end{vmatrix}.$$

161. Expand the determinant

$$\begin{vmatrix} 2 & 1 & 1 & x \\ 1 & 2 & 1 & y \\ 1 & 1 & 2 & z \\ 1 & 1 & 1 & t \end{vmatrix}$$

by the elements of the last column and evaluate.

162. Expand the determinant

$$\begin{vmatrix} a & 1 & 1 & 1 \\ b & 0 & 1 & 1 \\ c & 1 & 0 & 1 \\ d & 1 & 1 & 0 \end{vmatrix}$$

by the elements of the first column and evaluate.

Sec. 5. Computing Determinants

Compute the determinants:

***163.** $\begin{vmatrix} 13547 & 13647 \\ 28423 & 28523 \end{vmatrix}$. **164.** $\begin{vmatrix} 246 & 427 & 327 \\ 1014 & 543 & 443 \\ -342 & 721 & 621 \end{vmatrix}$.

165. $\begin{vmatrix} 3 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 \\ 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 3 \end{vmatrix}.$ **166.** $\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{vmatrix}.$ **167.** $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{vmatrix}$

168. $\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 16 \\ 1 & 8 & 27 & 64 \end{vmatrix}.$ **169.** $\begin{vmatrix} 1 & 2 & 3 & 4 \\ -2 & 1 & -4 & 3 \\ 3 & -4 & -1 & 2 \\ 4 & 3 & -2 & -1 \end{vmatrix}.$

170. $\begin{vmatrix} 2 & 1 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 & 1 \\ 1 & 1 & 4 & 1 & 1 \\ 1 & 1 & 1 & 5 & 1 \\ 1 & 1 & 1 & 1 & 6 \end{vmatrix}.$ **171.** $\begin{vmatrix} 5 & 6 & 0 & 0 & 0 \\ 1 & 5 & 6 & 0 & 0 \\ 0 & 1 & 5 & 6 & 0 \\ 0 & 0 & 1 & 5 & 6 \\ 0 & 0 & 0 & 1 & 5 \end{vmatrix}.$

172. $\begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & a & b \\ 1 & a & 0 & c \\ 1 & b & c & 0 \end{vmatrix}.$ **173.** $\begin{vmatrix} x & y & x+y \\ y & x+y & x \\ x+y & x & y \end{vmatrix}.$

174. $\begin{vmatrix} x & 0 & -1 & 1 & 0 \\ 1 & x & -1 & 1 & 0 \\ 1 & 0 & x-1 & 0 & 1 \\ 0 & 1 & -1 & x & 1 \\ 0 & 1 & -1 & 0 & x \end{vmatrix}.$ **175.** $\begin{vmatrix} 1+x & 1 & 1 & 1 \\ 1 & 1-x & 1 & 1 \\ 1 & 1 & 1+z & 1 \\ 1 & 1 & 1 & 1-z \end{vmatrix}.$

176. $\begin{vmatrix} 1 & 1 & 2 & 3 \\ 1 & 2-x^2 & 2 & 3 \\ 2 & 3 & 1 & 5 \\ 2 & 3 & 1 & 9-x^2 \end{vmatrix}.$

177. $\begin{vmatrix} \cos(a-b) & \cos(b-c) & \cos(c-a) \\ \cos(a+b) & \cos(b+c) & \cos(c+a) \\ \sin(a+b) & \sin(b+c) & \sin(c+a) \end{vmatrix}.$

178. $$\begin{vmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{vmatrix}.$$

***179.** $$\begin{vmatrix} 1 & 2 & 3 & \dots & n \\ -1 & 0 & 3 & \dots & n \\ -1 & -2 & 0 & \dots & n \\ \dots & \dots & \dots & \dots & \dots \\ -1 & -2 & -3 & \dots & 0 \end{vmatrix}.$$

***180.** $$\begin{vmatrix} 1 & a_1 & a_2 & & a_n \\ 1 & a_1+b_1 & a_2 & & a_n \\ 1 & a_1 & a_2+b_2 & \dots & a_n \\ \dots & \dots & \dots & \dots & \dots \\ 1 & a_1 & a_2 & \dots & a_n+b_n \end{vmatrix}.$$

***181.** $$\begin{vmatrix} 1 & x_1 & x_2 & \dots & x_{n-1} & x_n \\ 1 & x & x_2 & \dots & x_{n-1} & x_n \\ 1 & x_1 & x & \dots & x_{n-1} & x_n \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & x_1 & x_2 & \dots & x & x_n \\ 1 & x_1 & x_2 & \dots & x_{n-1} & x \end{vmatrix}.$$

***182.** $$\begin{vmatrix} 1 & 2 & 3 & \dots & n-1 & n \\ 1 & 3 & 3 & \dots & n-1 & n \\ 1 & 2 & 5 & \dots & n-1 & n \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & 2 & 3 & \dots & 2n-3 & n \\ 1 & 2 & 3 & \dots & n-1 & 2n-1 \end{vmatrix}.$$

***183.** $$\begin{vmatrix} 1 & 2 & 2 & \dots & 2 \\ 2 & 2 & 2 & \dots & 2 \\ 2 & 2 & 3 & \dots & 2 \\ \dots & \dots & \dots & \dots & \dots \\ 2 & 2 & 2 & \dots & n \end{vmatrix}.$$

***184.** $$\begin{vmatrix} 1 & b_1 & 0 & 0 & \dots & 0 & 0 \\ -1 & 1-b_1 & b_2 & 0 & \dots & 0 & 0 \\ 0 & -1 & 1-b_2 & b_3 & \dots & 0 & 0 \\ \hdotsfor{7} \\ 0 & 0 & 0 & 0 & \dots & 1-b_{n-1} & b_n \\ 0 & 0 & 0 & 0 & \dots & -1 & 1-b_n \end{vmatrix}.$$

***185.** $$\begin{vmatrix} a & a+h & a+2h & \dots & a+(n-1)h \\ -a & a & 0 & \dots & 0 \\ 0 & -a & a & \dots & 0 \\ \hdotsfor{5} \\ 0 & 0 & 0 & \dots & a \end{vmatrix}.$$

***186.** $$\begin{vmatrix} a & -(a+h) & \dots & (-1)^{n-1}[a+(n-1)h] \\ a & a & \dots & 0 \\ 0 & a & \dots & 0 \\ \hdotsfor{4} \\ 0 & 0 & \dots & a \end{vmatrix}.$$

***187.** $$\begin{vmatrix} 1 & C_n^1 & C_n^2 & C_n^3 & \dots & C_a^{n-2} & C_n^{n-1} & C_n^n \\ 1 & C_{n-1}^1 & C_{n-1}^2 & C_{n-1}^3 & \dots & C_{n-1}^{n-2} & C_{n-1}^{n-1} & 0 \\ 1 & C_{n-2}^1 & C_{n-2}^2 & C_{n-2}^3 & \dots & C_{n-2}^{n-2} & 0 & 0 \\ \hdotsfor{8} \\ 1 & C_2^1 & C_2^2 & 0 & \dots & 0 & 0 & 0 \\ 1 & C_1^1 & 0 & 0 & \dots & 0 & 0 & 0 \\ a_0 & a_1 & a_2 & a_3 & \dots & a_{n-2} & a_{n-1} & a_n \end{vmatrix}.$$

***188.** $$\begin{vmatrix} a_0 & -1 & 0 & \dots & 0 & 0 \\ a_1 & x & -1 & \dots & 0 & 0 \\ a_2 & 0 & x & \dots & 0 & 0 \\ \hdotsfor{6} \\ a_{n-1} & 0 & 0 & \dots & x & -1 \\ a_n & 0 & 0 & \dots & 0 & x \end{vmatrix}.$$

***189.** $$\begin{vmatrix} n & n-1 & n-2 & \dots & 3 & 2 & 1 \\ -1 & x & 0 & \dots & 0 & 0 & 0 \\ 0 & -1 & x & \dots & 0 & 0 & 0 \\ \hdotsfor{7} \\ 0 & 0 & 0 & \dots & -1 & x & 0 \\ 0 & 0 & 0 & \dots & 0 & -1 & x \end{vmatrix}.$$

***190.** Compute the difference $f(x+1)-f(x)$, where

$$f(x)=\begin{vmatrix} 1 & 0 & 0 & 0 & \dots & 0 & x \\ 1 & 2 & 0 & 0 & \dots & 0 & x^2 \\ 1 & 3 & 3 & 0 & \dots & 0 & x^3 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & n & C_n^2 & C_n^3 & \dots & C_n^{n-1} & x^n \\ 1 & n+1 & C_{n+1}^2 & C_{n+1}^3 & \dots & C_{n+1}^{n-1} & x^{n+1} \end{vmatrix}.$$

Compute the determinants:

***191.** $$\begin{vmatrix} x & a_1 & a_2 & \dots & a_{n-1} & 1 \\ a_1 & x & a_2 & \dots & a_{n-1} & 1 \\ a_1 & a_2 & x & \dots & a_{n-1} & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_1 & a_2 & a_3 & \dots & x & 1 \\ a_1 & a_2 & a_3 & \dots & a_n & 1 \end{vmatrix}.$$

***192.** $$\begin{vmatrix} x & a & a & \dots & a \\ a & x & a & \dots & a \\ a & a & x & \dots & a \\ \dots & \dots & \dots & \dots & \dots \\ a & a & a & \dots & x \end{vmatrix}.$$

193. $$\begin{vmatrix} x & a & a & \dots & a & a \\ -a & x & a & \dots & a & a \\ -a & -a & x & \dots & a & a \\ \dots & \dots & \dots & \dots & \dots & \dots \\ -a & -a & -a & \dots & -a & x \end{vmatrix}.$$

***194.** $$\begin{vmatrix} -a_1 & a_1 & 0 & \dots & 0 & 0 \\ 0 & -a_2 & a_2 & \dots & 0 & 0 \\ 0 & 0 & -a_3 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & -a_n & a_n \\ 1 & 1 & 1 & \dots & 1 & 1 \end{vmatrix}.$$

***195.** $$\begin{vmatrix} a_1 & -a_2 & 0 & \dots & 0 & 0 \\ 0 & a_2 & -a_3 & \dots & 0 & 0 \\ 0 & 0 & a_3 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & a_{n-1} & -a_n \\ 1 & 1 & 1 & \dots & 1 & 1+a_n \end{vmatrix}.$$

***196.** $$\begin{vmatrix} h & -1 & 0 & 0 & \dots & 0 \\ hx & h & -1 & 0 & \dots & 0 \\ hx^2 & hx & h & -1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ hx^n & hx^{n-1} & hx^{n-2} & hx^{n-3} & \dots & h \end{vmatrix}.$$

***197.** $$\begin{vmatrix} 0 & 1 & 1 & \dots & 1 & 1 \\ 1 & 0 & x & \dots & x & x \\ 1 & x & 0 & \dots & x & x \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & x & x & \dots & 0 & x \\ 1 & x & x & \dots & x & 0 \end{vmatrix}.$$

***198.** $$\begin{vmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & a_1+a_2 & \dots & a_1+a_n \\ 1 & a_2+a_1 & 0 & \dots & a_2+a_n \\ \dots & \dots & \dots & \dots & \dots \\ 1 & a_n+a_1 & a_n+a_2 & \dots & 0 \end{vmatrix}.$$

***199.** $$\begin{vmatrix} 1 & 2 & 3 & \dots & n-1 & n \\ 1 & 1 & 1 & \dots & 1 & 1-n \\ 1 & 1 & 1 & \dots & 1-n & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & 1-n & 1 & \dots & 1 & 1 \end{vmatrix}.$$

***200.** $$\begin{vmatrix} 2 & 1-\frac{1}{n} & 1-\frac{1}{n} & \dots & 1-\frac{1}{n} \\ 1-\frac{1}{n} & 2 & 1-\frac{1}{n} & \dots & 1-\frac{1}{n} \\ \dots & \dots & \dots & \dots & \dots \\ 1-\frac{1}{n} & 1-\frac{1}{n} & 1-\frac{1}{n} & \dots & 2 \end{vmatrix}$$

(order $n+1$).

***201.** $$\begin{vmatrix} 1 & a & a^2 & a^3 & \dots & a^n \\ x_{11} & 1 & a & a^2 & \dots & a^{n-1} \\ x_{21} & x_{22} & 1 & a & \dots & a^{n-2} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ x_{n1} & x_{n2} & x_{n3} & x_{n4} & \dots & 1 \end{vmatrix}.$$

***202.** $$\begin{vmatrix} 1 & 2 & 3 & 4 & \dots & n \\ 2 & 1 & 2 & 3 & \dots & n-1 \\ 3 & 2 & 1 & 2 & \dots & n-2 \\ 4 & 3 & 2 & 1 & \dots & n-3 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ n & n-1 & n-2 & n-3 & \dots & 1 \end{vmatrix}.$$

***203.** $$\begin{vmatrix} a_0 & b_1 & 0 & 0 & \dots & 0 & 0 \\ a_1 & -b_0 & b_2 & 0 & \dots & 0 & 0 \\ a_2 & 0 & -b_1 & b_3 & \dots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n-1} & 0 & 0 & 0 & \dots & -b_{n-2} & b_n \\ a_n & 0 & 0 & 0 & \dots & 0 & -b_{n-1} \end{vmatrix}.$$

***204.**

$$\begin{vmatrix} a & a^2 & 0 & 0 & \dots & 0 & 0 \\ 1 & 2a+b & (a+b)^2 & 0 & \dots & 0 & 0 \\ 0 & 1 & 2a+3b & (a+2b)^2 & \dots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \dots & 2a+(2n-1)b & (a+nb)^2 \\ 0 & 0 & 0 & 0 & \dots & 1 & 2a+(2n+1)b \end{vmatrix}.$$

***205.**

$$\begin{vmatrix} x & y & 0 & \dots & 0 & 0 \\ 0 & x & y & \dots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \dots & x & y \\ y & 0 & 0 & \dots & 0 & x \end{vmatrix}.$$

***206.**

$$\begin{vmatrix} 1+x_1y_1 & 1+x_1y_2 & \dots & 1+x_1y_n \\ 1+x_2y_1 & 1+x_2y_2 & \dots & 1+x_2y_n \\ \cdots & \cdots & \cdots & \cdots \\ 1+x_ny_1 & 1+x_ny_2 & \dots & 1+x_ny_n \end{vmatrix}.$$

207. $$\begin{vmatrix} a_1-b_1 & a_1-b_2 & \dots & a_1-b_n \\ a_2-b_1 & a_2-b_2 & \dots & a_2-b_n \\ \cdots & \cdots & \cdots & \cdots \\ a_n-b_1 & a_n-b_2 & \dots & a_n-b_n \end{vmatrix}.$$

***208.** $$\begin{vmatrix} 1+a_1+x_1 & a_1+x_2 & \dots & a_2+x_n \\ a_2+x_1 & 1+a_2+x_2 & \dots & a_2+x_n \\ \cdots & \cdots & \cdots & \cdots \\ a_n+x_1 & a_n+x_2 & \dots & 1+a_n+x_n \end{vmatrix}.$$

209. $$\begin{vmatrix} a^n-\alpha & a^{n+1}-\alpha & \dots & a^{n+p-1}-\alpha \\ a^{n+p}-\alpha & a^{n+p+1}-\alpha & \dots & a^{n+2p-1}-\alpha \\ \dots & \dots & \dots & \dots \\ a^{n+p(p-1)}-\alpha & a^{n+p(p-1)+1}-\alpha & \dots & a^{n+p^2-1}-\alpha \end{vmatrix}.$$

210. Prove that the determinant

$$\begin{vmatrix} f_1(a_1) & f_1(a_2) & \dots & f_1(a_n) \\ f_2(a_1) & f_2(a_2) & \dots & f_2(a_n) \\ \dots & \dots & \dots & \dots \\ f_n(a_1) & f_n(a_2) & \dots & f_n(a_n) \end{vmatrix}$$

is equal to zero if $f_1(x), f_2(x), \dots, f_n(x)$ are polynomials in x, each of degree not exceeding $n-2$, and the numbers $a_1, a_2, \dots, a_n$ are arbitrary.

Compute the determinants:

***211.** $$\begin{vmatrix} 1 & 2 & 3 & 4 & \dots & n-1 & n \\ -1 & x & 0 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & x & 0 \\ 0 & 0 & 0 & 0 & \dots & -1 & x \end{vmatrix}.$$

***212.** $$\begin{vmatrix} a_1+x_1 & a_2 & a_3 & \dots & a_{n-1} & a_n \\ -x_1 & x_2 & 0 & \dots & 0 & 0 \\ 0 & -x_2 & x_3 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & -x_{n-1} & x_n \end{vmatrix}.$$

***213.** $$\begin{vmatrix} a_0 & a_1 & a_2 & \dots & a_{n-1} & a_n \\ -y_1 & x_1 & 0 & \dots & 0 & 0 \\ 0 & -y_2 & x_2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & -y_n & x_n \end{vmatrix}.$$

***214.** $$\begin{vmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & a_1 & 0 & \dots & 0 \\ 1 & 0 & a_2 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 0 & 0 & \dots & a_n \end{vmatrix}.$$

***215.** $$\begin{vmatrix} n!\,a_0 & (n-1)!\,a_1 & (n-2)!\,a_2 & \dots & a_n \\ -n & x & 0 & \dots & 0 \\ 0 & -(n-1) & x & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & x \end{vmatrix}.$$

216. $$\begin{vmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & a_1 & 0 & 0 & 0 \\ 1 & 1 & a_2 & 0 & 0 \\ 1 & 0 & 1 & a_3 & 0 \\ 1 & 0 & 0 & 1 & a_4 \end{vmatrix}.$$ Write an nth-order determinant of this structure and compute it.

Compute the determinants:

217.

$$\begin{vmatrix} \alpha+\beta & \alpha\beta & 0 & \dots & 0 & 0 \\ 1 & \alpha+\beta & \alpha\beta & \dots & 0 & 0 \\ 0 & 1 & \alpha+\beta & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & \alpha+\beta \end{vmatrix}.$$

218.

$$\begin{vmatrix} 2 & 1 & 0 & 0 & \dots & 0 \\ 1 & 2 & 1 & 0 & \dots & 0 \\ 0 & 1 & 2 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 2 \end{vmatrix}.$$

***219.** $$\begin{vmatrix} 2\cos\theta & 1 & 0 & \dots & 0 & 0 \\ 1 & 2\cos\theta & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & 2\cos\theta \end{vmatrix}.$$

220. $$\begin{vmatrix} \cos\theta & 1 & 0 & \dots & 0 \\ 1 & 2\cos\theta & 1 & \dots & 0 \\ 0 & 1 & 2\cos\theta & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 2\cos\theta \end{vmatrix}.$$

***221.**

$$\begin{vmatrix} x & 1 & 0 & \dots & 0 \\ 1 & x & 1 & \dots & 0 \\ 0 & 1 & x & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & x \end{vmatrix}.$$

***222.**

$$\begin{vmatrix} x_1y_1 & x_1y_2 & x_1y_3 & \dots & x_1y_n \\ x_1y_2 & x_2y_2 & x_2y_3 & \dots & x_2y_n \\ x_1y_3 & x_2y_3 & x_3y_3 & \dots & x_3y_n \\ \dots & \dots & \dots & \dots & \dots \\ x_1y_n & x_2y_n & x_3y_n & \dots & x_ny_n \end{vmatrix}.$$

***223.** $$\begin{vmatrix} 1+a_1 & 1 & 1 & \dots & 1 \\ 1 & 1+a_2 & 1 & \dots & 1 \\ 1 & 1 & 1+a_3 & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & 1 & \dots & 1+a_n \end{vmatrix}.$$

224. $$\begin{vmatrix} 1 & 1 & & 1 & a_1+1 \\ 1 & 1 & \dots & a_2+1 & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & a_{n-1}+1 & \dots & 1 & 1 \\ a_n+1 & 1 & \dots & 1 & 1 \end{vmatrix}.$$

***225.** $$\begin{vmatrix} a_1 & x & x & \dots & x \\ x & a_2 & x & \dots & x \\ x & x & a_3 & \dots & x \\ \dots & \dots & \dots & \dots & \dots \\ x & x & x & \dots & a_n \end{vmatrix}.$$

***226.** $$\begin{vmatrix} x_1 & a_2 & a_3 & \dots & a_{n-1} & a_n \\ a_1 & x_2 & a_3 & \dots & a_{n-1} & a_n \\ a_1 & a_2 & x_3 & \dots & a_{n-1} & a_n \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_1 & a_2 & a_3 & \dots & x_{n-1} & a_n \\ a_1 & a_2 & a_3 & \dots & a_{n-1} & x_n \end{vmatrix}.$$

***227.** $$\begin{vmatrix} x_1 & a_2 b_1 & a_3 b_1 & \dots & a_n b_1 \\ a_1 b_2 & x_2 & a_3 b_2 & \dots & a_n b_2 \\ a_1 b_3 & a_2 b_3 & x_3 & \dots & a_n b_3 \\ \dots & \dots & \dots & \dots & \dots \\ a_1 b_n & a_2 b_n & a_3 b_n & \dots & x_n \end{vmatrix}.$$

***228.** $$\begin{vmatrix} x_1-m & x_2 & x_3 & \dots & x_n \\ x_1 & x_2-m & x_3 & \dots & x_n \\ x_1 & x_2 & x_3-m & \dots & x_n \\ \dots & \dots & \dots & \dots & \dots \\ x_1 & x_2 & x_3 & \dots & x_n-m \end{vmatrix}.$$

229. Solve the equation

$$\begin{vmatrix} a_1 & a_2 & \dots & a_{n-1} & a_n-\alpha_n x \\ a_1 & a_2 & \dots & a_{n-1}-\alpha_{n-1} x & a_n \\ \dots & \dots & \dots & \dots & \dots \\ a_1-\alpha_1 x & a_2 & \dots & a_{n-1} & a_n \end{vmatrix}=0.$$

Compute the determinants:

***230.** $$\begin{vmatrix} a & 0 & 0 & \dots & 0 & 0 & b \\ 0 & a & 0 & \dots & 0 & b & 0 \\ 0 & 0 & a & \dots & b & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & b & \dots & a & 0 & 0 \\ 0 & b & 0 & \dots & 0 & a & 0 \\ b & 0 & 0 & \dots & 0 & 0 & a \end{vmatrix} \quad \text{(of order } 2n\text{)}.$$

***231.**
$$\begin{vmatrix} 1 & -b & -b & -b & & -b \\ 1 & na & -2b & -3b & \dots & -(n-1)b \\ 1 & (n-1)a & a & -3b & \dots & -(n-1)b \\ 1 & (n-2)a & a & a & \dots & -(n-1)b \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & 2a & a & a & \dots & a \end{vmatrix}.$$

***232.**
$$\begin{vmatrix} (x-a_1)^2 & a_2^2 & & a_n^2 \\ a_1^2 & (x-a_2)^2 & \dots & a_n^2 \\ \dots & \dots & \dots & \dots \\ a_1^2 & a_2^2 & \dots & (x-a_n)^2 \end{vmatrix}.$$

***233.**
$$\begin{vmatrix} (x-a_1)^2 & a_1 a_2 & \dots & a_1 a_n \\ a_1 a_2 & (x-a_2)^2 & \dots & a_2 a_n \\ \dots & \dots & \dots & \dots \\ a_1 a_n & a_2 a_n & \dots & (x-a_n)^2 \end{vmatrix}.$$

***234.**
$$\begin{vmatrix} 1-b_1 & b_2 & 0 & 0 & \dots & 0 \\ -1 & 1-b_2 & b_3 & 0 & \dots & 0 \\ 0 & -1 & 1-b_3 & b_4 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 1-b_n \end{vmatrix}.$$

***235.**
$$\begin{vmatrix} 0 & a_2 & a_3 & a_4 & \dots & a_{n-1} & a_n \\ b_1 & 0 & a_3 & a_4 & \dots & a_{n-1} & a_n \\ b_1 & b_2 & 0 & a_4 & \dots & a_{n-1} & a_n \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ b_1 & b_2 & b_3 & b_4 & \dots & 0 & a_n \\ b_1 & b_2 & b_3 & b_4 & \dots & b_{n-1} & 0 \end{vmatrix}.$$

***236.**
$$\begin{vmatrix} 1 & 2 & 3 & 4 & 5 & \dots & n \\ 1 & 1 & 2 & 3 & 4 & \dots & n-1 \\ 1 & x & 1 & 2 & 3 & \dots & n-2 \\ 1 & x & x & 1 & 2 & \dots & n-3 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & x & x & x & x & \dots & 1 \end{vmatrix}.$$

***237.**
$$\begin{vmatrix} 1 & 2 & 3 & 4 & \dots & n \\ x & 1 & 2 & 3 & \dots & n-1 \\ x & x & 1 & 2 & \dots & n-2 \\ x & x & x & 1 & \dots & n-3 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ x & x & x & x & \dots & 1 \end{vmatrix}.$$

*238. $\begin{vmatrix} a_0 x^n & a_1 x^{n-1} & a_2 x^{n-2} & \dots & a_{n-1} x & a_n \\ a_0 x & b_1 & 0 & & 0 & 0 \\ a_0 x^2 & a_1 x & b_2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_0 x^{n-1} & a_1 x^{n-2} & a_2 x^{n-3} & \dots & b_{n-1} & 0 \\ a_0 x^n & a_1 x^{n-1} & a_2 x^{n-2} & \dots & a_{n-1} x & b_n \end{vmatrix}$.

***239.** Prove that the determinant

$$\begin{vmatrix} a_{00} x^n & a_{01} x^{n-1} & a_{02} x^{n-2} & \dots & a_{0n} \\ a_{10} x & a_{11} & 0 & & 0 \\ a_{20} x^2 & a_{21} x & a_{22} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ a_{n0} x^n & a_{n1} x^{n-1} & a_{n2} x^{n-2} & \dots & a_{nn} \end{vmatrix} = x^n \cdot \begin{vmatrix} a_{00} & a_{01} & a_{02} & \dots & a_{0n} \\ a_{10} & a_{11} & 0 & \dots & 0 \\ a_{20} & a_{21} & a_{22} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ a_{n0} & a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}.$$

Compute the determinants:

***240.**

$$\begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & C_2^1 & C_3^1 & \dots & C_n^1 \\ 1 & C_3^2 & C_4^2 & \dots & C_{n+1}^2 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & C_n^{n-1} & C_{n+1}^{n-1} & \dots & C_{2n-2}^{n-1} \end{vmatrix}.$$

***241.**

$$\begin{vmatrix} 1 & 1 & \dots & 1 \\ C_m^1 & C_{m+1}^1 & \dots & C_{m+n}^1 \\ C_{m+1}^2 & C_{m+2}^2 & \dots & C_{m+n+1}^2 \\ \dots & \dots & \dots & \dots \\ C_{m+n-1}^n & C_{m+n}^n & \dots & C_{m+2n-1}^n \end{vmatrix}.$$

***242.**

$$\begin{vmatrix} 1 & 1 & 0 & 0 & \dots & 0 \\ 1 & C_2^1 & C_2^2 & 0 & \dots & 0 \\ 1 & C_3^1 & C_3^2 & C_3^3 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & C_n^1 & C_n^2 & C_n^3 & \dots & C_n^{n-1} \end{vmatrix}.$$

***243.**

$$\begin{vmatrix} C_m^k & C_m^{k+1} & \dots & C_m^{k+n} \\ C_{m+1}^k & C_{m+1}^{k+1} & \dots & C_{m+1}^{k+n} \\ \dots & \dots & \dots & \dots \\ C_{m+n}^k & C_{m+n}^{k+1} & \dots & C_{m+n}^{k+n} \end{vmatrix}.$$

***244.** $\begin{vmatrix} C_{k+m}^m & C_{k+m+1}^m & \dots & C_{k+2m}^m \\ C_{k+m+1}^m & C_{k+m+2}^m & \dots & C_{k+2m+1}^m \\ \dots & \dots & \dots & \dots \\ C_{k+2m}^m & C_{k+2m+1}^m & \dots & C_{k+3m}^m \end{vmatrix}$.

***245.** $\begin{vmatrix} 1 & 0 & 0 & \dots & 0 & 1 \\ 1 & C_1^1 & 0 & \dots & 0 & x \\ 1 & C_2^1 & C_2^2 & \dots & 0 & x^2 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & C_n^1 & C_n^2 & \dots & C_n^{n-1} & x^n \end{vmatrix}$.

***246.**
$$\begin{vmatrix} 1 & 0 & 0 & 0 & \dots & 1 \\ 1 & 1! & 0 & 0 & \dots & x \\ 1 & 2 & 2! & 0 & \dots & x^2 \\ 1 & 3 & 3\cdot 2 & 3! & \dots & x^3 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & n & n(n-1) & n(n-1)(n-2) & \dots & x^n \end{vmatrix}.$$

***247.**
$$\begin{vmatrix} \alpha & \alpha+\delta & \alpha+2\delta & \alpha+3\delta & \dots & \alpha+(n-1)\delta \\ \alpha & 2\alpha+\delta & 3\alpha+3\delta & 4\alpha+6\delta & \dots & C_n^1\alpha+C_n^2\delta \\ \alpha & 3\alpha+\delta & 6\alpha+4\delta & 10\alpha+10\delta & \dots & C_{n+1}^2\alpha+C_{n+1}^3\delta \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \alpha & C_n^{n-1}\alpha+\delta & C_{n+1}^{n-1}\alpha+C_{n+1}^n\delta & C_{n+2}^{n-1}\alpha+C_{n+2}^n\delta & \dots & C_{2n-2}^{n-1}\alpha+C_{2n-2}^n\delta \end{vmatrix}.$$

***248.**
$$\begin{vmatrix} x & y & y & \dots & y & y \\ z & x & y & \dots & y & y \\ z & z & x & \dots & y & y \\ \dots & \dots & \dots & \dots & \dots & \dots \\ z & z & z & \dots & x & y \\ z & z & z & \dots & z & x \end{vmatrix}.$$

***249.**
$$\begin{vmatrix} a & a & a & \dots & a & 0 \\ a & a & a & \dots & 0 & b \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a & 0 & b & \dots & b & b \\ 0 & b & b & \dots & b & b \end{vmatrix}.$$

250.
$$\begin{vmatrix} a_1 & x & x & \dots & x \\ y & a_2 & x & \dots & x \\ \dots & \dots & \dots & \dots & \dots \\ y & y & y & \dots & a_n \end{vmatrix}.$$

251.
$$\begin{vmatrix} c_1 & a & a & \dots & a & 1 \\ b & c_2 & a & \dots & a & 1 \\ b & b & c_3 & \dots & a & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ b & b & b & \dots & c_n & 1 \\ 1 & 1 & 1 & \dots & 1 & 0 \end{vmatrix}.$$

***252.**
$$\begin{vmatrix} \lambda & a & a & a & \dots & a \\ b & \alpha & \beta & \beta & \dots & \beta \\ b & \beta & \alpha & \beta & \dots & \beta \\ b & \beta & \beta & \alpha & \dots & \beta \\ \dots & \dots & \dots & \dots & \dots & \dots \\ b & \beta & \beta & \beta & \dots & \alpha \end{vmatrix}.$$

***253.**
$$\begin{vmatrix} 1 & 2 & 3 & \dots & n \\ 2 & 3 & 4 & \dots & 1 \\ 3 & 4 & 5 & \dots & 2 \\ \dots & \dots & \dots & \dots & \dots \\ n & 1 & 2 & \dots & n-1 \end{vmatrix}.$$

*254. $$\begin{vmatrix} a & a+h & a+2h & \dots & a+(n-1)h \\ a+h & a+2h & a+3h & \dots & a \\ a+2h & a+3h & a+4h & \dots & a+h \\ \dots & \dots & \dots & \dots & \dots \\ a+(n-1)h & a & a+h & \dots & a+(n-2)h \end{vmatrix}.$$

255. $$\begin{vmatrix} 1 & x & x^2 & \dots & x^{n-1} \\ x^{n-1} & 1 & x & \dots & x^{n-2} \\ \dots & \dots & \dots & \dots & \dots \\ x & x^2 & x^3 & \dots & 1 \end{vmatrix}.$$

*256. $$\begin{vmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{vmatrix}.$$

257. $$\begin{vmatrix} a & b & c & d & e & f & g & h \\ b & a & d & c & f & e & h & g \\ c & d & a & b & g & h & e & f \\ d & c & b & a & h & g & f & e \\ e & f & g & h & a & b & c & d \\ f & e & h & g & b & a & d & c \\ g & h & e & f & c & d & a & b \\ h & g & f & e & d & c & b & a \end{vmatrix}.$$

*258. $$\begin{vmatrix} x & a_1 & a_2 & \dots & a_n \\ a_1 & x & a_2 & \dots & a_n \\ \dots & \dots & \dots & \dots & \dots \\ a_1 & a_2 & a_3 & \dots & x \end{vmatrix}.$$

*259. $$\begin{vmatrix} \cos^{n-1}\varphi_1 & \cos^{n-2}\varphi_1 & \dots & \cos\varphi_1 & 1 \\ \cos^{n-1}\varphi_2 & \cos^{n-2}\varphi_2 & \dots & \cos\varphi_2 & 1 \\ \dots & \dots & \dots & \dots & \dots \\ \cos^{n-1}\varphi_n & \cos^{n-2}\varphi_n & \dots & \cos\varphi_n & 1 \end{vmatrix}.$$

260. $$\begin{vmatrix} 1 & 1 & \dots & 1 \\ \sin\varphi_1 & \sin\varphi_2 & \dots & \sin\varphi_n \\ \sin^2\varphi_1 & \sin^2\varphi_2 & \dots & \sin^2\varphi_n \\ \dots & \dots & \dots & \dots \\ \sin^{n-1}\varphi_1 & \sin^{n-1}\varphi_2 & \dots & \sin^{n-1}\varphi_n \end{vmatrix}.$$

261. $$\begin{vmatrix} a^n & (a-1)^n & \dots & (a-n)^n \\ a^{n-1} & (a-1)^{n-1} & \dots & (a-n)^{n-1} \\ \dots & \dots & \dots & \dots \\ a & a-1 & \dots & a-n \\ 1 & 1 & \dots & 1 \end{vmatrix}.$$

262. $$\begin{vmatrix} (a_1+x)^n & (a_1+x)^{n-1} & \dots & a_1+x & 1 \\ (a_2+x)^n & (a_2+x)^{n-1} & \dots & a_2+x & 1 \\ \dots & \dots & \dots & \dots & \dots \\ (a_{n+1}+x)^n & (a_{n+1}+x)^{n-1} & \dots & a_{n+1}+x & 1 \end{vmatrix}.$$

263. $$\begin{vmatrix} (2n-1)^n & (2n-2)^n & \dots & n^n & (2n)^n \\ (2n-1)^{n-1} & (2n-2)^{n-1} & \dots & n^{n-1} & (2n)^{n-1} \\ \dots & \dots & \dots & \dots & \dots \\ 2n-1 & 2n-2 & \dots & n & 2n \\ 1 & 1 & \dots & 1 & 1 \end{vmatrix}.$$

***264.** $$\begin{vmatrix} w_1 & a_1 & a_1^2 & \dots & a_1^{n-1} \\ w_2 & a_2 & a_2^2 & \dots & a_2^{n-1} \\ \dots & \dots & \dots & \dots & \dots \\ w_n & a_n & a_n^2 & \dots & a_n^{n-1} \end{vmatrix}.$$

***265.**

$$\begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ x_1+1 & x_2+1 & x_3+1 & \dots & x_n+1 \\ x_1^2+x_1 & x_2^2+x_2 & x_3^2+x_3 & \dots & x_n^2+x_n \\ \dots & \dots & \dots & \dots & \dots \\ x_1^{n-1}+x_1^{n-2} & x_2^{n-1}+x_2^{n-2} & x_3^{n-1}+x_3^{n-2} & \dots & x_n^{n-1}+x_n^{n-2} \end{vmatrix}.$$

266.

$$\begin{vmatrix} 1 & 1 & \dots & 1 \\ 1+\sin\varphi_1 & 1+\sin\varphi_2 & \dots & 1+\sin\varphi_n \\ \sin\varphi_1+\sin^2\varphi_1 & \sin\varphi_2+\sin^2\varphi_2 & \dots & \sin\varphi_n+\sin^2\varphi_n \\ \dots & \dots & \dots & \dots \\ \sin^{n-2}\varphi_1+\sin^{n-1}\varphi_1 & \sin^{n-2}\varphi_2+\sin^{n-1}\varphi_2 & \dots & \sin^{n-2}\varphi_n+\sin^{n-1}\varphi_n \end{vmatrix}.$$

267. $$\begin{vmatrix} 1 & 1 & \dots & 1 \\ \varphi_1(x_1) & \varphi_1(x_2) & \dots & \varphi_1(x_n) \\ \varphi_2(x_1) & \varphi_2(x_2) & \dots & \varphi_2(x_n) \\ \dots & \dots & \dots & \dots \\ \varphi_{n-1}(x_1) & \varphi_{n-1}(x_2) & \dots & \varphi_{n-1}(x_n) \end{vmatrix}$$

where $\varphi_k(x)=x^k+a_{1k}x^{k-1}+\dots+a_{kk}$.

268. $$\begin{vmatrix} 1 & 1 & \dots & 1 \\ F_1(\cos\varphi_1) & F_1(\cos\varphi_2) & \dots & F_1(\cos\varphi_n) \\ F_2(\cos\varphi_1) & F_2(\cos\varphi_2) & \dots & F_2(\cos\varphi_n) \\ \dots & \dots & \dots & \dots \\ F_{n-1}(\cos\varphi_1) & F_{n-1}(\cos\varphi_2) & \dots & F_{n-1}(\cos\varphi_n) \end{vmatrix}$$

where $F_k(x) = a_{0k}x^k + a_{1k}x^{k-1} + \ldots + a_{kk}$.

***269.**

$$\begin{vmatrix} 1 & 1 & \ldots & 1 \\ \binom{x_1}{1} & \binom{x_2}{1} & \ldots & \binom{x_n}{1} \\ \binom{x_1}{2} & \binom{x_2}{2} & \ldots & \binom{x_n}{2} \\ \ldots & \ldots & \ldots & \ldots \\ \binom{x_1}{n-1} & \binom{x_2}{n-1} & \ldots & \binom{x_n}{n-1} \end{vmatrix}$$

where $\binom{x}{k} = \frac{x(x-1)\ldots(x-k+1)}{1\cdot 2\ldots k}$.

***270.** Prove that the value of the determinant

$$\begin{vmatrix} 1 & a_1 & a_1^2 & \ldots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \ldots & a_2^{n-1} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & a_n & a_n^2 & \ldots & a_n^{n-1} \end{vmatrix}$$

is divisible by $1^{n-1}2^{n-2}\ldots(n-1)$ for integral $a_1, a_2, \ldots, a_n$.

Compute the determinants:

***271.**

$$\begin{vmatrix} 1 & 2 & 3 & \ldots & n \\ 1 & 2^3 & 3^3 & \ldots & n^3 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & 2^{2n-1} & 3^{2n-1} & \ldots & n^{2n-1} \end{vmatrix}.$$

***272.**

$$\begin{vmatrix} \frac{x_1}{x_1-1} & \frac{x_2}{x_2-1} & \ldots & \frac{x_n}{x_n-1} \\ x_1 & x_2 & \ldots & x_n \\ x_1^2 & x_2^2 & \ldots & x_n^2 \\ \ldots & \ldots & \ldots & \ldots \\ x_1^{n-1} & x_2^{n-1} & \ldots & x_n^{n-1} \end{vmatrix}.$$

***273.**

$$\begin{vmatrix} a_1^n & a_1^{n-1}b_1 & a_1^{n-2}b_1^2 & \ldots & a_1b_1^{n-1} & b_1^n \\ a_2^n & a_2^{n-1}b_2 & a_2^{n-2}b_2^2 & \ldots & a_2b_2^{n-1} & b_2^n \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{n+1}^n & a_{n+1}^{n-1}b_{n+1} & a_{n+1}^{n-2}b_{n+1}^2 & \ldots & a_{n+1}b_{n+1}^{n-1} & b_{n+1}^n \end{vmatrix}.$$

274.

$$\begin{vmatrix} \sin^{n-1}\alpha_1 & \sin^{n-2}\alpha_1\cos\alpha_1 & \ldots & \sin\alpha_1\cos^{n-2}\alpha_1 & \cos^{n-1}\alpha_1 \\ \sin^{n-1}\alpha_2 & \sin^{n-2}\alpha_2\cos\alpha_2 & \ldots & \sin\alpha_2\cos^{n-2}\alpha_2 & \cos^{n-1}\alpha_2 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \sin^{n-1}\alpha_n & \sin^{n-2}\alpha_n\cos\alpha_n & \ldots & \sin\alpha_n\cos^{n-2}\alpha_n & \cos^{n-1}\alpha_n \end{vmatrix}.$$

***275.**

$$\begin{vmatrix} a_1^{2n}+1 & a_1^{2n-1}+a_1 & a_1^{2n-2}+a_1^2 & \dots & a_1^{n+1}+a_1^{n-1} & a_1^n \\ a_2^{2n}+1 & a_2^{2n-1}+a_2 & a_2^{2n-2}+a_2^2 & \dots & a_2^{n+1}+a_2^{n-1} & a_2^n \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n+1}^{2n}+1 & a_{n+1}^{2n-1}+a_{n+1} & a_{n+1}^{2n-2}+a_{n+1}^2 & \dots & a_{n+1}^{n+1}+a_{n+1}^{n-1} & a_{n+1}^n \end{vmatrix}.$$

***276.** $$\begin{vmatrix} 1 & \cos\varphi_0 & \cos 2\varphi_0 & \dots & \cos(n-1)\varphi_0 \\ 1 & \cos\varphi_1 & \cos 2\varphi_1 & \dots & \cos(n-1)\varphi_1 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & \cos\varphi_{n-1} & \cos 2\varphi_{n-1} & \dots & \cos(n-1)\varphi_{n-1} \end{vmatrix}.$$

***277.** $$\begin{vmatrix} \sin(n+1)\alpha_0 & \sin n\alpha_0 & \dots & \sin\alpha_0 \\ \sin(n+1)\alpha_1 & \sin n\alpha_1 & \dots & \sin\alpha_1 \\ \dots & \dots & \dots & \dots \\ \sin(n+1)\alpha_n & \sin n\alpha_n & \dots & \sin\alpha_n \end{vmatrix}.$$

***278.** $$\begin{vmatrix} 1 & 1 & \dots & 1 \\ x_1(x_1-1) & x_2(x_2-1) & \dots & x_n(x_n-1) \\ x_1^2(x_1-1) & x_2^2(x_2-1) & \dots & x_n^2(x_n-1) \\ \dots & \dots & \dots & \dots \\ x_1^{n-1}(x_1-1) & x_2^{n-1}(x_2-1) & \dots & x_n^{n-1}(x_n-1) \end{vmatrix}.$$

***279.** $$\begin{vmatrix} 1 & 1 & \dots & 1 \\ x_1^2 & x_2^2 & \dots & x_n^2 \\ x_1^3 & x_2^3 & \dots & x_n^3 \\ \dots & \dots & \dots & \dots \\ x_1^n & x_2^n & \dots & x_n^n \end{vmatrix}.$$

***280.** $$\begin{vmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_n \\ x_1^2 & x_2^2 & \dots & x_n^2 \\ \dots & \dots & \dots & \dots \\ x_1^{n-2} & x_2^{n-2} & \dots & x_n^{n-2} \\ x_1^n & x_2^n & \dots & x_n^n \end{vmatrix}.$$

281.

$$\begin{vmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_n \\ x_1^2 & x_2^2 & \dots & x_n^2 \\ \dots & \dots & \dots & \dots \\ x_1^{s-1} & x_2^{s-1} & \dots & x_n^{s-1} \\ x_1^{s+1} & x_2^{s+1} & \dots & x_n^{s+1} \\ \dots & \dots & \dots & \dots \\ x_1^n & x_2^n & \dots & x_n^n \end{vmatrix}.$$

***282.**

$$\begin{vmatrix} 1+x_1 & 1+x_1^2 & \dots & 1+x_1^n \\ 1+x_2 & 1+x_2^2 & \dots & 1+x_2^n \\ \dots & \dots & \dots & \dots \\ 1+x_n & 1+x_n^2 & \dots & 1+x_n^n \end{vmatrix}.$$

283.

$$\begin{vmatrix} 1 & x & x^2 & x^3 \\ x^3 & x^2 & x & 1 \\ 1 & 2x & 3x^2 & 4x^3 \\ 4x^3 & 3x^2 & 2x & 1 \end{vmatrix}.$$

284.

$$\begin{vmatrix} 1 & x & x^2 & x^3 & x^4 \\ 1 & 2x & 3x^2 & 4x^3 & 5x^4 \\ 1 & 4x & 9x^2 & 16x^3 & 25x^4 \\ 1 & y & y^2 & y^3 & y^4 \\ 1 & 2y & 3y^2 & 4y^3 & 5y^4 \end{vmatrix}.$$

***285.**

$$\begin{vmatrix} 1 & x & x^2 & \dots & x^n \\ 1 & 2x & 3x^2 & \dots & (n+1)x^n \\ 1 & 2^2 x & 3^2 x^2 & \dots & (n+1)^2 x^n \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 2^{n-1} x & 3^{n-1} x^2 & \dots & (n+1)^{n-1} x^n \\ 1 & y & y^2 & \dots & y^n \end{vmatrix}.$$

***286.**

$$\begin{vmatrix} 1 & x & x^2 & \dots & x^{n-1} \\ 1 & 2x & 3x^2 & \dots & nx^{n-1} \\ 1 & 2^2 x & 3^2 x^2 & \dots & n^2 x^{n-1} \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 2^{k-1} x & 3^{k-1} x^2 & \dots & n^{k-1} x^{n-1} \\ 1 & y_1 & y_1^2 & \dots & y_1^{n-1} \\ 1 & y_2 & y_2^2 & \dots & y_2^{n-1} \\ \dots & \dots & \dots & \dots & \dots \\ 1 & y_{n-k} & y_{n-k}^2 & \dots & y_{n-k}^{n-1} \end{vmatrix}.$$

***287.**

$$\begin{vmatrix} 1 & x & x^2 & \dots & x^{n-1} \\ 0 & 1 & C_2^1 x & \dots & C_{n-1}^1 x^{n-2} \\ 0 & 0 & 1 & \dots & C_{n-1}^2 x^{n-3} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & C_{n-1}^{k-1} x^{n-k} \\ 1 & y & y^2 & \dots & y^{n-1} \\ 0 & 1 & C_2^1 y & \dots & C_{n-1}^1 y^{n-2} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & C_{n-1}^{n-k-1} y^k \end{vmatrix}.$$

288.

(a) Write the expansion of a fourth-order determinant in terms of the minors of the first two rows.

(b) Compute the determinant

$$\begin{vmatrix} 1 & 2 & 2 & 1 \\ 0 & 1 & 0 & 2 \\ 2 & 0 & 1 & 1 \\ 0 & 2 & 0 & 1 \end{vmatrix}$$

using the expansion by minors of the second order.

(c) Compute the determinant

$$\begin{vmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{vmatrix}$$

via an expansion by second-order minors.

(d) Compute the determinant of Problem 145.

Compute the determinants:

(e) $$\begin{vmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 2 & 3 & 4 & 0 & 0 & 0 \\ 3 & 6 & 10 & 0 & 0 & 0 \\ 4 & 9 & 14 & 1 & 1 & 1 \\ 5 & 15 & 24 & 1 & 5 & 9 \\ 9 & 24 & 38 & 1 & 25 & 81 \end{vmatrix},$$ (f) $$\begin{vmatrix} a_1 & 0 & b_1 & 0 \\ 0 & c_1 & 0 & d_1 \\ b_2 & 0 & a_2 & 0 \\ 0 & d_2 & 0 & c_2 \end{vmatrix},$$

(g) $$\begin{vmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ x_1 & x_2 & 0 & 0 & 0 & x_3 \\ a_1 & b_1 & 1 & 1 & 1 & c_1 \\ a_2 & b_2 & x_1 & x_2 & x_3 & c_2 \\ a_3 & b_3 & x_1^2 & x_2^2 & x_3^2 & c_3 \\ x_1^2 & x_2^2 & 0 & 0 & 0 & x_3^2 \end{vmatrix},$$ (h) $$\begin{vmatrix} \lambda & 0 & 0 & \dots & 0 & a \\ x_1 & \alpha & \beta & \dots & \beta & y_1 \\ x_2 & \beta & \alpha & \dots & \beta & y_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ x_n & \beta & \beta & \dots & \alpha & y_n \\ a & 0 & 0 & \dots & 0 & \lambda \end{vmatrix}.$$

(i) Compute the determinant of Problem 230 using the Laplace theorem.

(j) Compute the determinant of Problem 171 using the Laplace theorem.

(k) Compute the determinant

$$\begin{vmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & x_1 & x_2 & x_3 & x_4 \\ 0 & x_1^2 & x_2^2 & x_3^2 & x_4^2 \end{vmatrix}.$$

(l) Let A, B, C and D be third-order determinants formed from the array

$$\begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{pmatrix}$$

by deleting the first, second, third and fourth columns, respectively. Prove that

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 & 0 & 0 \\ a_2 & b_2 & c_2 & d_2 & 0 & 0 \\ a_3 & b_3 & c_3 & d_3 & 0 & 0 \\ 0 & 0 & a_1 & b_1 & c_1 & d_1 \\ 0 & 0 & a_2 & b_2 & c_2 & d_2 \\ 0 & 0 & a_3 & b_3 & c_3 & d_3 \end{vmatrix} = AD - BC.$$

(*m) Compute the fifteenth-order determinant

$$\begin{vmatrix} \Delta & \Delta_1 & \Delta_1 \\ \Delta_1 & \Delta & \Delta_1 \\ \Delta_1 & \Delta_1 & \Delta \end{vmatrix}$$

formed (as indicated) from the following blocks:

$$\Delta = \begin{pmatrix} a & x & x & -x & -x \\ x & 2a & a & 0 & 0 \\ x & a & 2a & 0 & 0 \\ -x & 0 & 0 & 2a & a \\ -x & 0 & 0 & a & 2a \end{pmatrix}, \quad \Delta_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}.$$

Sec. 6. Multiplication of Determinants

289. Using the rule for multiplying matrices, represent the following products of determinants in the form of a determinant:

(a) $\begin{vmatrix} 4 & 3 \\ 1 & 3 \end{vmatrix} \cdot \begin{vmatrix} 1 & -2 \\ -3 & 2 \end{vmatrix}$,

(b) $\begin{vmatrix} 3 & 2 & 5 \\ -1 & 3 & 6 \\ 1 & -1 & 2 \end{vmatrix} \cdot \begin{vmatrix} -2 & 3 & 4 \\ -1 & -3 & 5 \\ 2 & 1 & -1 \end{vmatrix}$,

(c) $\begin{vmatrix} 2 & 1 & 1 & 1 \\ -1 & 2 & 1 & 1 \\ -1 & -1 & 2 & 1 \\ -1 & -1 & -1 & 2 \end{vmatrix} \cdot \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} \cdot \begin{vmatrix} -2 & 1 \\ -1 & 2 \end{vmatrix}$.

290. Compute the determinant Δ by multiplying it by the determinant δ:

(a) $\Delta = \begin{vmatrix} 1 & 2 & 3 & 4 \\ -1 & 0 & -3 & -8 \\ -1 & 1 & 0 & -13 \\ 2 & 3 & 5 & 15 \end{vmatrix}$,

$\delta = \begin{vmatrix} 1 & -2 & -3 & -11 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix}$;

(b) $\Delta = \begin{vmatrix} -1 & -9 & -2 & 3 \\ -5 & 5 & 3 & -2 \\ -12 & -6 & 1 & 1 \\ 9 & 0 & -2 & 1 \end{vmatrix}$, $\delta = \begin{vmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ -3 & 4 & 2 & 1 \end{vmatrix}$;

(c) $\Delta = \begin{vmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{vmatrix}$, $\delta = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{vmatrix}$.

291. Compute the square of the determinant:

$$\text{(a)} \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{vmatrix}, \quad \text{(b)} \begin{vmatrix} 1 & -1 & 1 & -1 \\ 2 & 2 & 1 & 1 \\ 2 & 0 & -3 & -1 \\ 3 & -7 & -1 & 9 \end{vmatrix},$$

$$\text{(c)} \begin{vmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{vmatrix}.$$

292. The determinant

$$\begin{vmatrix} a_{00} & a_{01} & a_{02} & \dots & a_{0,\,n-1} \\ a_{10} & a_{11} & a_{12} & \dots & a_{1,\,n-1} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n-1,\,0} & a_{n-1,\,1} & a_{n-1,\,2} & \dots & a_{n-1,\,n-1} \end{vmatrix} = D.$$

What is

$$\begin{vmatrix} \varphi_0(x_1) & \varphi_0(x_2) & \dots & \varphi_0(x_n) \\ \varphi_1(x_1) & \varphi_1(x_2) & \dots & \varphi_1(x_n) \\ \dots & \dots & \dots & \dots \\ \varphi_{n-1}(x_1) & \varphi_{n-1}(x_2) & \dots & \varphi_{n-1}(x_n) \end{vmatrix}$$

where $\varphi_i(x) = a_{0i} + a_{1i}x + \dots + a_{n-1,\,i}\, x^{n-1}$?

Use the result obtained to find the solution of Problems 265, 267, 268.

Compute the determinants:

***293.**

$$\text{(a)} \begin{vmatrix} (b_0+a_0)^n & (b_1+a_0)^n & \dots & (b_n+a_0)^n \\ (b_0+a_1)^n & (b_1+a_1)^n & \dots & (b_n+a_1)^n \\ \dots & \dots & \dots & \dots \\ (b_0+a_n)^n & (b_1+a_n)^n & \dots & (b_n+a_n)^n \end{vmatrix},$$

$$\text{(b)} \begin{vmatrix} \dfrac{1-\alpha_1^n\beta_1^n}{1-\alpha_1\beta_1} & \dfrac{1-\alpha_1^n\beta_2^n}{1-\alpha_1\beta_2} & \dots & \dfrac{1-\alpha_1^n\beta_n^n}{1-\alpha_1\beta_n} \\ \dfrac{1-\alpha_2^n\beta_1^n}{1-\alpha_2\beta_1} & \dfrac{1-\alpha_2^n\beta_2^n}{1-\alpha_2\beta_2} & \dots & \dfrac{1-\alpha_2^n\beta_n^n}{1-\alpha_2\beta_n} \\ \dots & \dots & \dots & \dots \\ \dfrac{1-\alpha_n^n\beta_1^n}{1-\alpha_n\beta_1} & \dfrac{1-\alpha_n^n\beta_2^n}{1-\alpha_n\beta_2} & \dots & \dfrac{1-\alpha_n^n\beta_n^n}{1-\alpha_n\beta_n} \end{vmatrix}.$$

***294.** $$\begin{vmatrix} \sin 2\alpha_1 & \sin(\alpha_1+\alpha_2) & \dots & \sin(\alpha_1+\alpha_n) \\ \sin(\alpha_2+\alpha_1) & \sin 2\alpha_2 & \dots & \sin(\alpha_2+\alpha_n) \\ \dots & \dots & \dots & \dots \\ \sin(\alpha_n+\alpha_1) & \sin(\alpha_n+\alpha_2) & \dots & \sin 2\alpha_n \end{vmatrix}.$$

***295.** $$\begin{vmatrix} s_0 & s_1 & s_2 & \dots & s_{n-1} & 1 \\ s_1 & s_2 & s_3 & \dots & s_n & x \\ \dots & \dots & \dots & \dots & \dots & \dots \\ s_{n-1} & s_n & s_{n+1} & \dots & s_{2n-2} & x^{n-1} \\ s_n & s_{n+1} & s_{n+2} & \dots & s_{2n-1} & x^n \end{vmatrix}$$

where $s_k = x_1^k + x_2^k + \dots + x_n^k$.

***296.** $$\begin{vmatrix} a & b & c & d & l & m & n & p \\ b & -a & -d & -c & m & -l & p & -n \\ c & d & -a & -b & n & -p & -l & m \\ d & -c & b & -a & p & n & -m & -l \\ l & -m & -n & -p & -a & b & c & d \\ m & l & p & -n & -b & -a & d & -c \\ n & -p & l & m & -c & -d & -a & b \\ p & n & -m & l & -d & c & -b & -a \end{vmatrix}.$$

***297.** $$\begin{vmatrix} \cos\varphi & \sin\varphi & \cos\varphi & \sin\varphi \\ \cos 2\varphi & \sin 2\varphi & 2\cos 2\varphi & 2\sin 2\varphi \\ \cos 3\varphi & \sin 3\varphi & 3\cos 3\varphi & 3\sin 3\varphi \\ \cos 4\varphi & \sin 4\varphi & 4\cos 4\varphi & 4\sin 4\varphi \end{vmatrix}.$$

***298.**
$$\begin{vmatrix} \cos n\varphi & n\cos n\varphi & \sin n\varphi & n\sin n\varphi \\ \cos(n+1)\varphi & (n+1)\cos(n+1)\varphi & \sin(n+1)\varphi & (n+1)\sin(n+1)\varphi \\ \cos(n+2)\varphi & (n+2)\cos(n+2)\varphi & \sin(n+2)\varphi & (n+2)\sin(n+2)\varphi \\ \cos(n+3)\varphi & (n+3)\cos(n+3)\varphi & \sin(n+3)\varphi & (n+3)\sin(n+3)\varphi \end{vmatrix}.$$

***299.**
$$\begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \varepsilon & \varepsilon^2 & \dots & \varepsilon^{n-1} \\ 1 & \varepsilon^2 & \varepsilon^4 & \dots & \varepsilon^{2n-2} \\ \dots & \dots & \dots & \dots & \dots \\ 1 & \varepsilon^{n-1} & \varepsilon^{2(n-1)} & \dots & \varepsilon^{(n-1)^2} \end{vmatrix},$$

where $\varepsilon = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$.

***300.**
$$\begin{vmatrix} a_0 & a_1 & a_2 & \dots & a_{n-1} \\ a_{n-1} & a_0 & a_1 & \dots & a_{n-2} \\ \dots & \dots & \dots & \dots & \dots \\ a_1 & a_2 & a_3 & \dots & a_0 \end{vmatrix}$$

(cyclic determinant).

301. Apply the result of Problem 300 to the determinant

$$\begin{vmatrix} x & u & z & y \\ y & x & u & z \\ z & y & x & u \\ u & z & y & x \end{vmatrix}.$$

302. Apply the result of Problem 300 to Problems 192, 205, and 255.

Compute the determinants:

303. $$\begin{vmatrix} 1 & C_{n-1}^{1} & C_{n-1}^{2} & \cdots & C_{n-1}^{n-2} & 1 \\ 1 & 1 & C_{n-1}^{1} & \cdots & C_{n-1}^{n-3} & C_{n-1}^{n-2} \\ C_{n-1}^{n-2} & 1 & 1 & \cdots & C_{n-1}^{n-4} & C_{n-1}^{n-3} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ C_{n-1}^{1} & C_{n-1}^{2} & C_{n-1}^{3} & \cdots & 1 & 1 \end{vmatrix}.$$

304. $$\begin{vmatrix} 1 & 2a & 3a^2 & \cdots & na^{n-1} \\ na^{n-1} & 1 & 2a & \cdots & (n-1)\,a^{n-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 2a & 3a^2 & 4a^3 & \cdots & 1 \end{vmatrix}.$$

305. $$\begin{vmatrix} s-a_1 & s-a_2 & \cdots & s-a_n \\ s-a_n & s-a_1 & \cdots & s-a_{n-1} \\ \cdots & \cdots & \cdots & \cdots \\ s-a_2 & s-a_3 & \cdots & s-a_1 \end{vmatrix}$$

where $s=a_1+a_2+\ldots+a_n$.

306. $$\begin{vmatrix} t^{n-1} & C_n^1 t^{n-2} & C_n^2 t^{n-3} & \cdots & C_n^{n-2} t & C_n^{n-1} \\ C_n^{n-1} & t^{n-1} & C_n^1 t^{n-2} & \cdots & C_n^{n-3} t^2 & C_n^{n-2} t \\ C_n^{n-2} t & C_n^{n-1} & t^{n-1} & \cdots & C_n^{n-4} t^3 & C_n^{n-3} t^2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ C_n^1 t^{n-2} & C_n^2 t^{n-3} & C_n^3 t^{n-4} & \cdots & C_n^{n-1} & t^{n-1} \end{vmatrix}.$$

307. $$\begin{array}{c} \overbrace{\qquad\qquad\qquad\qquad\qquad}^{p}\qquad\qquad\overbrace{\qquad\qquad\qquad\qquad}^{n-p} \\ \begin{vmatrix} -1 & -1 & \cdots & -1 & -1 & 1 & 1 & \cdots & 1 \\ 1 & -1 & \cdots & -1 & -1 & -1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & -1 & -1 & -1 & -1 & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -1 & -1 & \cdots & -1 & 1 & 1 & 1 & \cdots & -1 \end{vmatrix} \end{array}.$$

***308.** $$\begin{vmatrix} \cos\frac{\pi}{n} & \cos\frac{2\pi}{n} & \cdots & -1 \\ -1 & \cos\frac{\pi}{n} & \cdots & \cos\frac{(n-1)\pi}{n} \\ \cos\frac{(n-1)\pi}{n} & -1 & \cdots & \cos\frac{(n-2)\pi}{n} \\ \cdots & \cdots & \cdots & \cdots \\ \cos\frac{2\pi}{n} & \cos\frac{3\pi}{n} & \cdots & \cos\frac{\pi}{n} \end{vmatrix}.$$

309. $$\begin{vmatrix} \cos\theta & \cos 2\theta & \cdots & \cos n\theta \\ \cos n\theta & \cos\theta & \cdots & \cos(n-1)\theta \\ \cdots & \cdots & \cdots & \cdots \\ \cos 2\theta & \cos 3\theta & \cdots & \cos\theta \end{vmatrix}.$$

310.

$$\begin{vmatrix} \sin a & \sin(a+h) & \sin(a+2h) & \cdots & \sin[a+(n-1)h] \\ \sin[a+(n-1)h] & \sin a & \sin(a+h) & \cdots & \sin[a+(n-2)h] \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \sin(a+h) & \sin(a+2h) & \sin(a+3h) & \cdots & \sin a \end{vmatrix}.$$

***311.** $$\begin{vmatrix} 1^2 & 2^2 & 3^2 & \cdots & n^2 \\ n^2 & 1^2 & 2^2 & \cdots & (n-1)^2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 2^2 & 3^2 & 4^2 & \cdots & 1^2 \end{vmatrix}.$$

312. Prove that

$$\begin{vmatrix} a_0 & a_1 & a_1 & a_2 & a_1 & a_2 & a_2 \\ a_2 & a_0 & a_1 & a_1 & a_2 & a_1 & a_2 \\ a_2 & a_2 & a_0 & a_1 & a_1 & a_2 & a_1 \\ a_1 & a_2 & a_2 & a_0 & a_1 & a_1 & a_2 \\ a_2 & a_1 & a_2 & a_2 & a_0 & a_1 & a_1 \\ a_1 & a_2 & a_1 & a_2 & a_2 & a_0 & a_1 \\ a_1 & a_1 & a_2 & a_1 & a_2 & a_2 & a_0 \end{vmatrix}$$
$$=(a_0+3a_1+3a_2)(a_0^2-a_0a_1-a_0a_2+2a_1^2+2a_2^2-3a_1a_2)^3.$$

313. Compute the determinant

$$\begin{vmatrix} a_1 & a_2 & a_3 & \dots & a_n \\ -a_n & a_1 & a_2 & \dots & a_{n-1} \\ -a_{n-1} & -a_n & a_1 & \dots & a_{n-2} \\ \dots & \dots & \dots & \dots & \dots \\ -a_2 & -a_3 & -a_4 & \dots & a_1 \end{vmatrix}$$

(skew-symmetric determinant).

***314.** Prove that a cyclic determinant of order $2n$ may be represented as a product of a cyclic determinant of order n and a skew-cyclic determinant of order n.

315. Compute the determinant

$$\begin{vmatrix} a_1 & a_2 & a_3 & \dots & a_n \\ \mu a_n & a_1 & a_2 & \dots & a_{n-1} \\ \mu a_{n-1} & \mu a_n & a_1 & \dots & a_{n-2} \\ \dots & \dots & \dots & \dots & \dots \\ \mu a_2 & \mu a_3 & \mu a_4 & \dots & a_1 \end{vmatrix}.$$

Sec. 7. Miscellaneous Problems

316. Prove that if

$$\Delta(x) = \begin{vmatrix} a_{11}(x) & a_{12}(x) & \dots & a_{1n}(x) \\ a_{21}(x) & a_{22}(x) & \dots & a_{2n}(x) \\ \dots & \dots & \dots & \dots \\ a_{n1}(x) & a_{n2}(x) & \dots & a_{nn}(x) \end{vmatrix}$$

then

$$\Delta'(x) = \begin{vmatrix} a'_{11}(x) & a'_{12}(x) & \dots & a'_{1n}(x) \\ a_{21}(x) & a_{22}(x) & \dots & a_{2n}(x) \\ \dots & \dots & \dots & \dots \\ a_{n1}(x) & a_{n2}(x) & \dots & a_{nn}(x) \end{vmatrix}$$

$$+ \dots + \begin{vmatrix} a_{11}(x) & a_{12}(x) & \dots & a_{1n}(x) \\ a_{21}(x) & a_{22}(x) & \dots & a_{2n}(x) \\ \dots & \dots & \dots & \dots \\ a'_{n1}(x) & a'_{n2}(x) & \dots & a'_{nn}(x) \end{vmatrix}.$$

317. Prove that

$$\begin{vmatrix} a_{11}+x & a_{12}+x & \dots & a_{1n}+x \\ a_{21}+x & a_{22}+x & \dots & a_{2n}+x \\ \dots & \dots & \dots & \dots \\ a_{n1}+x & a_{n2}+x & \dots & a_{nn}+x \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} + x\sum_{k=1}^{n}\sum_{i=1}^{n} A_{ik}$$

where A_{ik} is the cofactor of the element a_{ik}.

318. Using the result of Problem 317, compute the determinants of Problems 200, 223, 224, 225, 226, 227, 228, 232, 233, 248, 249, 250.

319. Prove that the sum of the cofactors of all the elements of the determinant

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

is equal to

$$\begin{vmatrix} 1 & 1 & \dots & 1 \\ a_{21}-a_{11} & a_{22}-a_{12} & \dots & a_{2n}-a_{1n} \\ \dots & \dots & \dots & \dots \\ a_{n1}-a_{n-1,1} & a_{n2}-a_{n-1,2} & \dots & a_{nn}-a_{n-1,n} \end{vmatrix}.$$

Prove the following theorems:

320. The sum of the cofactors of all elements of a determinant remains unaltered if the same number is added to all elements.

321. If all the elements of one row (column) of a determinant are equal to unity, the sum of the cofactors of all elements of the determinant is equal to the determinant itself.

322. Compute the sum of the cofactors of all the elements of the determinant of Problem 250.

***323.** Compute the determinant

$$\begin{vmatrix} (a_1+b_1)^{-1} & (a_1+b_2)^{-1} & \dots & (a_1+b_n)^{-1} \\ (a_2+b_1)^{-1} & (a_2+b_2)^{-1} & \dots & (a_2+b_n)^{-1} \\ \dots & \dots & \dots & \dots \\ (a_n+b_1)^{-1} & (a_n+b_2)^{-1} & \dots & (a_n+b_n)^{-1} \end{vmatrix}.$$

324. Denote by P_n and Q_n the determinants

$$\begin{vmatrix} a_0 & 1 & 0 & \cdots & 0 & 0 \\ -1 & a_1 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & a_{n-2} & 1 \\ 0 & 0 & 0 & \cdots & -1 & a_{n-1} \end{vmatrix}$$

and

$$\begin{vmatrix} a_1 & 1 & 0 & \cdots & 0 & 0 \\ -1 & a_2 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & a_{n-2} & 1 \\ 0 & 0 & 0 & \cdots & -1 & a_{n-1} \end{vmatrix}$$

respectively, and prove that

$$\frac{P_n}{Q_n} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots + \cfrac{1}{a_{n-1}}}}}.$$

Compute the determinants

***325.**

$$\begin{vmatrix} c & a & 0 & \cdots & 0 & 0 \\ b & c & a & \cdots & 0 & 0 \\ 0 & b & c & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & c & a \\ 0 & 0 & 0 & \cdots & b & c \end{vmatrix}.$$

326.

$$\begin{vmatrix} p & q & 0 & \cdots & 0 & 0 \\ 2 & p & q & \cdots & 0 & 0 \\ 0 & 1 & p & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & p & q \\ 0 & 0 & 0 & \cdots & 1 & p \end{vmatrix}.$$

***327.** Represent the determinant

$$\begin{vmatrix} a_{11}+x & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}+x & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}+x \end{vmatrix}$$

in the form of a polynomial in powers of x.

***328.** Compute a determinant of order $(2n-1)$ in which the first $n-1$ elements of the principal diagonal are equal to unity and the other elements of the principal diagonal are equal to n. In each of the first $n-1$ rows, the n elements to the right of the principal diagonal are equal to unity and in each of the last n rows, the elements to the left of the principal diagonal are $n-1$, $n-2$, ..., 1. The other elements of the determinant are zero.

For example,
$$\begin{vmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 0 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 1 & 2 & 3 \end{vmatrix}; \quad \begin{vmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 4 & 0 & 0 \\ 0 & 0 & 1 & 2 & 3 & 4 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 & 4 \end{vmatrix}.$$

Compute the determinants:

***329.**
$$\begin{vmatrix} x & 1 & 0 & 0 & \cdots & 0 & 0 \\ -n & x-2 & 2 & 0 & \cdots & 0 & 0 \\ 0 & -(n-1) & x-4 & 3 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & -1 & x-2n \end{vmatrix}.$$

330.
$$\begin{vmatrix} x & 1 & 0 & 0 & \cdots & 0 & 0 \\ n-1 & x & 2 & 0 & \cdots & 0 & 0 \\ 0 & n-2 & x & 3 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & x \end{vmatrix}.$$

331.
$$\begin{vmatrix} x & a & 0 & 0 & \cdots & 0 & 0 \\ n(a-1) & x-1 & 2a & 0 & \cdots & 0 & 0 \\ 0 & (n-1)(a-1) & x-2 & 3a & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & a-1 & x-n \end{vmatrix}.$$

332.
$$\begin{vmatrix} 1^{n-1} & 2^{n-1} & \cdots & n^{n-1} \\ 2^{n-1} & 3^{n-1} & \cdots & (n+1)^{n-1} \\ \cdots & \cdots & \cdots & \cdots \\ n^{n-1} & (n+1)^{n-1} & \cdots & (2n-1)^{n-1} \end{vmatrix}.$$

333.
$$\begin{vmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \dots & \frac{1}{n} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \dots & \frac{1}{n+1} \\ \dots & \dots & \dots & \dots & \dots \\ \frac{1}{n} & \frac{1}{n+1} & \frac{1}{n+2} & \dots & \frac{1}{2n-1} \end{vmatrix}.$$

334. Find the coefficient of the lowest power of x in the determinant

$$\begin{vmatrix} (1+x)^{a_1 b_1} & (1+x)^{a_1 b_2} & \dots & (1+x)^{a_1 b_n} \\ (1+x)^{a_2 b_1} & (1+x)^{a_2 b_2} & \dots & (1+x)^{a_2 b_n} \\ \dots & \dots & \dots & \dots \\ (1+x)^{a_n b_1} & (1+x)^{a_n b_2} & \dots & (1+x)^{a_n b_n} \end{vmatrix}.$$

CHAPTER 3
SYSTEMS OF LINEAR EQUATIONS

Sec. 1. Cramer's Theorem

Solve the following systems of equations:

335. $$\begin{aligned} 2x_1 - x_2 - x_3 &= 4, \\ 3x_1 + 4x_2 - 2x_3 &= 11, \\ 3x_1 - 2x_2 + 4x_3 &= 11. \end{aligned}$$

336. $$\begin{aligned} x_1 + x_2 + 2x_3 &= -1, \\ 2x_1 - x_2 + 2x_3 &= -4, \\ 4x_1 + x_2 + 4x_3 &= -2. \end{aligned}$$

337. $$\begin{aligned} 3x_1 + 2x_2 + x_3 &= 5, \\ 2x_1 + 3x_2 + x_3 &= 1, \\ 2x_1 + x_2 + 3x_3 &= 11. \end{aligned}$$

338. $$\begin{aligned} x_1 + 2x_2 + 4x_3 &= 31, \\ 5x_1 + x_2 + 2x_3 &= 29, \\ 3x_1 - x_2 + x_3 &= 10. \end{aligned}$$

339. $$\begin{aligned} x_1 + x_2 + 2x_3 + 3x_4 &= 1, \\ 3x_1 - x_2 - x_3 - 2x_4 &= -4, \\ 2x_1 + 3x_2 - x_3 - x_4 &= -6, \\ x_1 + 2x_2 + 3x_3 - x_4 &= -4. \end{aligned}$$

340. $$\begin{aligned} x_1 + 2x_2 + 3x_3 - 2x_4 &= 6, \\ 2x_1 - x_2 - 2x_3 - 3x_4 &= 8, \\ 3x_1 + 2x_2 - x_3 + 2x_4 &= 4, \\ 2x_1 - 3x_2 + 2x_3 + x_4 &= -8. \end{aligned}$$

341. $$\begin{aligned} x_1 + 2x_2 + 3x_3 + 4x_4 &= 5, \\ 2x_1 + x_2 + 2x_3 + 3x_4 &= 1, \\ 3x_1 + 2x_2 + x_3 + 2x_4 &= 1, \\ 4x_1 + 3x_2 + 2x_3 + x_4 &= -5. \end{aligned}$$

342. $$\begin{aligned} x_2 - 3x_3 + 4x_4 &= -5, \\ x_1 \qquad - 2x_3 + 3x_4 &= -4, \\ 3x_1 + 2x_2 \qquad - 5x_4 &= 12, \\ 4x_1 + 3x_2 - 5x_3 \qquad &= 5. \end{aligned}$$

343.
$$\begin{aligned} 2x_1-x_2+3x_3+2x_4&=4,\\ 3x_1+3x_2+3x_3+2x_4&=6,\\ 3x_1-x_2-x_3+2x_4&=6,\\ 3x_1-x_2+3x_3-x_4&=6. \end{aligned}$$

344.
$$\begin{aligned} x_1+x_2+x_3+x_4&=0,\\ x_1+2x_2+3x_3+4x_4&=0,\\ x_1+3x_2+6x_3+10x_4&=0,\\ x_1+4x_2+10x_3+20x_4&=0. \end{aligned}$$

345.
$$\begin{aligned} x_1+3x_2+5x_3+7x_4&=12,\\ 3x_1+5x_2+7x_3+x_4&=0,\\ 5x_1+7x_2+x_3+3x_4&=4,\\ 7x_1+x_2+3x_3+5x_4&=16. \end{aligned}$$

346.
$$\begin{aligned} x_1+x_2+x_3+x_4+x_5&=0,\\ x_1-x_2+2x_3-2x_4+3x_5&=0,\\ x_1+x_2+4x_3+4x_4+9x_5&=0,\\ x_1-x_2+8x_3-8x_4+27x_5&=0,\\ x_1+x_2+16x_3+16x_4+81x_5&=0. \end{aligned}$$

347.
$$\begin{aligned} x_1+2x_2+3x_3+4x_4&=0,\\ x_1+x_2+2x_3+3x_4&=0,\\ x_1+5x_2+x_3+2x_4&=0,\\ x_1+5x_2+5x_3+2x_4&=0. \end{aligned}$$

348.
$$\begin{aligned} x_1+x_2+x_3+x_4&=0,\\ x_2+x_3+x_4+x_5&=0,\\ x_1+2x_2+3x_3&=2,\\ x_2+2x_3+3x_4&=-2,\\ x_3+2x_4+3x_5&=2. \end{aligned}$$

349.
$$\begin{aligned} x_1+4x_2+6x_3+4x_4+x_5&=0,\\ x_1+x_2+4x_3+6x_4+4x_5&=0,\\ 4x_1+x_2+x_3+4x_4+6x_5&=0,\\ 6x_1+4x_2+x_3+x_4+4x_5&=0,\\ 4x_1+6x_2+4x_3+x_4+x_5&=0. \end{aligned}$$

350.
$$\begin{aligned} 2x_1+x_2+x_3+x_4+x_5&=2,\\ x_1+2x_2+x_3+x_4+x_5&=0,\\ x_1+x_2+3x_3+x_4+x_5&=3,\\ x_1+x_2+x_3+4x_4+x_5&=-2,\\ x_1+x_2+x_3+x_4+5x_5&=5. \end{aligned}$$

351.
$$\begin{aligned} x_1+2x_2+3x_3+4x_4+5x_5&=13,\\ 2x_1+x_2+2x_3+3x_4+4x_5&=10,\\ 2x_1+2x_2+x_3+2x_4+3x_5&=11,\\ 2x_1+2x_2+2x_3+x_4+2x_5&=6,\\ 2x_1+2x_2+2x_3+2x_4+x_5&=3. \end{aligned}$$

352.
$$\begin{aligned} x_1+2x_2-3x_3+4x_4-x_5&=-1,\\ 2x_1-x_2+3x_3-4x_4+2x_5&=8,\\ 3x_1+x_2-x_3+2x_4-x_5&=3,\\ 4x_1+3x_2+4x_3+2x_4+2x_5&=-2,\\ x_1-x_2-x_3+2x_4-3x_5&=-3. \end{aligned}$$

353.
$$\begin{aligned} 2x_1-3x_2+4x_3-3x_4&=0,\\ 3x_1-x_2+11x_3-13x_4&=0,\\ 4x_1+5x_2-7x_3-2x_4&=0,\\ 13x_1-25x_2+x_3+11x_4&=0. \end{aligned}$$

Verify that the system has the solution $x_1=x_2=x_3=x_4=1$ and compute the determinant of the system.

354. Prove that the system

$$\begin{aligned} ax+by+cz+dt&=0,\\ bx-ay+dz-ct&=0,\\ cx-dy-az+bt&=0,\\ dx+cy-bz-at&=0 \end{aligned}$$

has a unique solution if a, b, c, d are real and not all zero.

Solve the following systems of equations:

355.
$$\begin{aligned} \alpha x_1+\alpha x_2+\ldots+\alpha x_{n-1}+\beta x_n&=a_n,\\ \alpha x_1+\alpha x_2+\ldots+\beta x_{n-1}+\alpha x_n&=a_{n-1},\\ &\cdots\cdots\cdots\cdots\\ \beta x_1+\alpha x_2+\ldots+\alpha x_{n-1}+\alpha x_n&=a_1 \end{aligned}$$

where $\alpha\neq\beta$.

356. $$\begin{aligned} &\frac{x_1}{b_1-\beta_1}+\frac{x_2}{b_1-\beta_2}+\cdots+\frac{x_n}{b_1-\beta_n}=1,\\ &\frac{x_1}{b_2-\beta_1}+\frac{x_2}{b_2-\beta_2}+\cdots+\frac{x_n}{b_2-\beta_n}=1,\\ &\dots\dots\dots\dots\dots\dots\\ &\frac{x_1}{b_n-\beta_1}+\frac{x_2}{b_n-\beta_2}+\cdots+\frac{x_n}{b_n-\beta_n}=1, \end{aligned}$$

where $b_1, b_2, \ldots, b_n, \beta_1, \beta_2, \ldots, \beta_n$ are all distinct.

357. $$\begin{aligned} &x_1+x_2+\ldots+x_n=1,\\ &x_1\alpha_1+x_2\alpha_2+\ldots+x_n\alpha_n=t,\\ &\dots\dots\dots\dots\dots\dots\\ &x_1\alpha_1^{n-1}+x_2\alpha_2^{n-1}+\ldots+x_n\alpha_n^{n-1}=t^{n-1} \end{aligned}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are all distinct.

358. $$\begin{aligned} &x_1+x_2\alpha_1+\cdots+x_n\alpha_1^{n-1}=u_1,\\ &x_1+x_2\alpha_2+\cdots+x_n\alpha_2^{n-1}=u_2,\\ &\dots\dots\dots\dots\dots\dots\\ &x_1+x_2\alpha_n+\cdots+x_n\alpha_n^{n-1}=u_n \end{aligned}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are all distinct.

359. $$\begin{aligned} &x_1+x_2+\ldots+x_n=u_1,\\ &x_1\alpha_1+x_2\alpha_2+\ldots+x_n\alpha_n=u_2,\\ &\dots\dots\dots\dots\dots\dots\\ &x_1\alpha_1^{n-1}+x_2\alpha_2^{n-1}+\ldots+x_n\alpha_n^{n-1}=u_n \end{aligned}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are all distinct.

360. $$\begin{aligned} &1+x_1+x_2+\ldots+x_n=0,\\ &1+2x_1+2^2x_2+\ldots+2^n x_n=0,\\ &\dots\dots\dots\dots\dots\dots\\ &1+nx_1+n^2x_2+\ldots+n^n x_n=0. \end{aligned}$$

Sec. 2. Rank of a Matrix

361. How many kth-order determinants can be formed from a matrix with m rows and n columns?

362. Form a matrix with rank equal to (a) 2, (b) 3.

363. Prove that the rank of a matrix remains unaltered if:
(a) rows and columns are interchanged;
(b) the elements of a row or column are multiplied by a non-zero number;
(c) two rows or two columns are interchanged;
(d) multiples of the elements of one row (column) are added to elements of another row (column).

364. The sum of two matrices having the same number of rows and columns is a matrix whose elements are the sums of the corresponding elements of the matrices being added. Prove that the rank of the sum of two matrices does not exceed the sum of the ranks of the matrices added.

365. How is the rank of a matrix affected by adjoining (a) one column, (b) two columns?

Compute the rank of the following matrices:

366.
$$\begin{pmatrix} 0 & 4 & 10 & 1 \\ 4 & 8 & 18 & 7 \\ 10 & 18 & 40 & 17 \\ 1 & 7 & 17 & 3 \end{pmatrix}.$$

367.
$$\begin{pmatrix} 75 & 0 & 116 & 39 & 0 \\ 171 & -69 & 402 & 123 & 45 \\ 301 & 0 & 87 & -417 & -169 \\ 114 & -46 & 268 & 82 & 30 \end{pmatrix}.$$

368.
$$\begin{pmatrix} 2 & 1 & 11 & 2 \\ 1 & 0 & 4 & -1 \\ 11 & 4 & 56 & 5 \\ 2 & -1 & 5 & -6 \end{pmatrix}.$$

369.
$$\begin{pmatrix} 14 & 12 & 6 & 8 & 2 \\ 6 & 104 & 21 & 9 & 17 \\ 7 & 6 & 3 & 4 & 1 \\ 35 & 30 & 15 & 20 & 5 \end{pmatrix}.$$

370.
$$\begin{pmatrix} 1 & 0 & 0 & 1 & 4 \\ 0 & 1 & 0 & 2 & 5 \\ 0 & 0 & 1 & 3 & 6 \\ 1 & 2 & 3 & 14 & 32 \\ 4 & 5 & 6 & 32 & 77 \end{pmatrix}.$$

371.
$$\begin{pmatrix} 1 & -2 & 3 & -1 & -1 & -2 \\ 2 & -1 & 1 & 0 & -2 & -2 \\ -2 & -5 & 8 & -4 & 3 & -1 \\ 6 & 0 & -1 & 2 & -7 & -5 \\ -1 & -1 & 1 & -1 & 2 & 1 \end{pmatrix}.$$

372.
$$\begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 \\ 1 & 1 & 4 & 1 \\ 1 & 1 & 1 & 5 \\ 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

373.
$$\begin{pmatrix} 1 & -1 & 2 & 3 & 4 \\ 2 & 1 & -1 & 2 & 0 \\ -1 & 2 & 1 & 1 & 3 \\ 1 & 5 & -8 & -5 & -12 \\ 3 & -7 & 8 & 9 & 13 \end{pmatrix}.$$

374.
$$\begin{pmatrix} 2 & 1 & 3 & -1 \\ 3 & -1 & 2 & 0 \\ 1 & 3 & 4 & -2 \\ 4 & -3 & 1 & 1 \end{pmatrix}.$$

375.
$$\begin{pmatrix} 3 & 2 & -1 & 2 & 0 & 1 \\ 4 & 1 & 0 & -3 & 0 & 2 \\ 2 & -1 & -2 & 1 & 1 & -3 \\ 3 & 1 & 3 & -9 & -1 & 6 \\ 3 & -1 & -5 & 7 & 2 & -7 \end{pmatrix}.$$

376.
$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 3 & 4 & 5 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 \end{pmatrix}.$$

377.
$$\begin{pmatrix} 1 & -1 & 2 & 0 & 0 & 1 \\ 0 & 1 & -1 & 2 & 0 & 1 \\ 1 & 0 & -1 & 0 & 2 & 1 \\ 1 & -1 & 0 & 0 & 1 & 2 \\ 2 & 0 & 0 & 1 & -1 & 1 \\ -1 & 1 & 0 & 1 & 1 & 2 \end{pmatrix}.$$

378.
$$\begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix}.$$

379.
$$\begin{pmatrix} 2 & 0 & 2 & 0 & 2 \\ 0 & 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix}.$$

380.
$$\begin{pmatrix} 2 & -1 & 1 & 3 & 4 \\ 2 & -1 & 2 & 1 & -2 \\ 2 & -3 & 1 & 2 & -2 \\ 1 & 0 & 1 & -2 & -6 \\ 1 & 2 & 1 & -1 & 0 \\ 4 & -1 & 3 & -1 & -8 \end{pmatrix}.$$

Sec. 3. Systems of Linear Forms

381. (a) Write two independent linear forms.
(b) Write three independent linear forms.

382. Form a system of four linear forms in five variables so that two of them are independent and the others are linear combinations of them.

Find the basic dependences between the forms of the system:

383.
$$\begin{aligned} y_1 &= 2x_1+2x_2+7x_3-x_4,\\ y_2 &= 3x_1-x_2+2x_3+4x_4,\\ y_3 &= x_1+x_2+3x_3+x_4. \end{aligned}$$

384.
$$\begin{aligned} y_1 &= 3x_1+2x_2-5x_3+4x_4,\\ y_2 &= 3x_1-x_2+3x_3-3x_4,\\ y_3 &= 3x_1+5x_2-13x_3+11x_4. \end{aligned}$$

385.
$$\begin{aligned} y_1 &= 2x_1+3x_2-4x_3-x_4,\\ y_2 &= x_1-2x_2+x_3+3x_4,\\ y_3 &= 5x_1-3x_2-x_3+8x_4,\\ y_4 &= 3x_1+8x_2-9x_3-5x_4. \end{aligned}$$

386.
$$\begin{aligned} y_1 &= 2x_1+x_2-x_3+x_4,\\ y_2 &= x_1+2x_2+x_3-x_4,\\ y_3 &= x_1+x_2+2x_3+x_4. \end{aligned}$$

387.
$$\begin{aligned} y_1 &= x_1+2x_2+3x_3+x_4,\\ y_2 &= 2x_1+3x_2+x_3+2x_4,\\ y_3 &= 3x_1+x_2+2x_3-2x_4,\\ y_4 &= 4x_2+2x_3+5x_4. \end{aligned}$$

388.
$$\begin{aligned} y_1 &= 2x_1+x_2,\\ y_2 &= 3x_1+2x_2,\\ y_3 &= x_1+x_2,\\ y_4 &= 2x_1+3x_2. \end{aligned}$$

389.
$$\begin{aligned} y_1 &= x_1+x_2+x_3+x_4+x_5,\\ y_2 &= x_1+2x_2+3x_3+4x_4+x_5,\\ y_3 &= x_1+3x_2+6x_3+10x_4+x_5,\\ y_4 &= x_1+4x_2+10x_3+20x_4+x_5. \end{aligned}$$

390.
$$\begin{aligned} y_1 &= x_1+2x_2+3x_3-4x_4,\\ y_2 &= 2x_1-x_2+2x_3+5x_4,\\ y_3 &= 2x_1-x_2+5x_3-4x_4,\\ y_4 &= 2x_1+3x_2-4x_3+x_4. \end{aligned}$$

391.
$$\begin{aligned} y_1 &= 2x_1+x_2-3x_3,\\ y_2 &= 3x_1+x_2-5x_3,\\ y_3 &= 4x_1+2x_2-x_3,\\ y_4 &= x_1-7x_3. \end{aligned}$$

392. $$\begin{aligned} y_1&=2x_1+3x_2+5x_3-4x_4+x_5,\\ y_2&=x_1-x_2+2x_3+3x_4+5x_5,\\ y_3&=3x_1+7x_2+8x_3-11x_4-3x_5,\\ y_4&=x_1-x_2+x_3-2x_4+3x_5. \end{aligned}$$

393. $$\begin{aligned} y_1&=2x_1-x_2+3x_3+4x_4-x_5,\\ y_2&=x_1+2x_2-3x_3+x_4+2x_5,\\ y_3&=5x_1-5x_2+12x_3+11x_4-5x_5,\\ y_4&=x_1-3x_2+6x_3+3x_4-3x_5. \end{aligned}$$

394. $$\begin{aligned} y_1&=x_1+2x_2+x_3-2x_4+x_5,\\ y_2&=2x_1-x_2+x_3+3x_4+2x_5,\\ y_3&=x_1-x_2+2x_3-x_4+3x_5,\\ y_4&=2x_1+x_2-3x_3+x_4-2x_5,\\ y_5&=x_1-x_2+3x_3-x_4+7x_5. \end{aligned}$$

395. $$\begin{aligned} y_1&=4x_1+3x_2-x_3+x_4-x_5,\\ y_2&=2x_1+x_2-3x_3+2x_4-5x_5,\\ y_3&=x_1-3x_2+x_4-2x_5,\\ y_4&=x_1+5x_2+2x_3-2x_4+6x_5. \end{aligned}$$

396. $$\begin{aligned} y_1&=x_1+2x_2-x_3+3x_4-x_5+2x_6,\\ y_2&=2x_1-x_2+3x_3-4x_4+x_5-x_6,\\ y_3&=3x_1+x_2-x_3+2x_4+x_5+3x_6,\\ y_4&=4x_1-7x_2+8x_3-15x_4+6x_5-5x_6,\\ y_5&=5x_1+5x_2-6x_3+11x_4+9x_6. \end{aligned}$$

397. $$\begin{aligned} y_1&=x_1+2x_2+x_3-3x_4+2x_5,\\ y_2&=2x_1+x_2+x_3+x_4-3x_5,\\ y_3&=x_1+x_2+2x_3+2x_4-2x_5,\\ y_4&=2x_1+3x_2-5x_3-17x_4+\lambda x_5. \end{aligned}$$

Choose λ so that the fourth form is a linear combination of the other three.

Sec. 4. Systems of Linear Equations

398. Solve the system of equations

$$\begin{aligned} x_1-2x_2+x_3+x_4&=1,\\ x_1-2x_2+x_3-x_4&=-1,\\ x_1-2x_2+x_3+5x_4&=5. \end{aligned}$$

399. Choose λ so that the following system of equations has a solution:

$$\begin{aligned} 2x_1 - x_2 + x_3 + x_4 &= 1,\\ x_1 + 2x_2 - x_3 + 4x_4 &= 2,\\ x_1 + 7x_2 - 4x_3 + 11x_4 &= \lambda. \end{aligned}$$

Solve the systems of equations:

400.

$$\begin{aligned} x_1 + x_2 - 3x_3 &= -1,\\ 2x_1 + x_2 - 2x_3 &= 1,\\ x_1 + x_2 + x_3 &= 3,\\ x_1 + 2x_2 - 3x_3 &= 1. \end{aligned}$$

401.

$$\begin{aligned} 2x_1 + x_2 + x_3 &= 2,\\ x_1 + 3x_2 + x_3 &= 5,\\ x_1 + x_2 + 5x_3 &= -7,\\ 2x_1 + 3x_2 - 3x_3 &= 14. \end{aligned}$$

402.

$$\begin{aligned} 2x_1 - x_2 + 3x_3 &= 3,\\ 3x_1 + x_2 - 5x_3 &= 0,\\ 4x_1 - x_2 + x_3 &= 3,\\ x_1 + 3x_2 - 13x_3 &= -6. \end{aligned}$$

403.

$$\begin{aligned} x_1 + 3x_2 + 2x_3 &= 0,\\ 2x_1 - x_2 + 3x_3 &= 0,\\ 3x_1 - 5x_2 + 4x_3 &= 0,\\ x_1 + 17x_2 + 4x_3 &= 0. \end{aligned}$$

404.

$$\begin{aligned} 2x_1 + x_2 - x_3 + x_4 &= 1,\\ 3x_1 - 2x_2 + 2x_3 - 3x_4 &= 2,\\ 5x_1 + x_2 - x_3 + 2x_4 &= -1,\\ 2x_1 - x_2 + x_3 - 3x_4 &= 4. \end{aligned}$$

405.

$$\begin{aligned} 2x_1 - x_2 + x_3 - x_4 &= 1,\\ 2x_1 - x_2 - 3x_4 &= 2,\\ 3x_1 - x_3 + x_4 &= -3,\\ 2x_1 + 2x_2 - 2x_3 + 5x_4 &= -6. \end{aligned}$$

406.

$$\begin{aligned} x_1 - 2x_2 + 3x_3 - 4x_4 &= 4,\\ x_2 - x_3 + x_4 &= -3,\\ x_1 + 3x_2 - 3x_4 &= 1,\\ -7x_2 + 3x_3 + x_4 &= -3. \end{aligned}$$

407.

$$\begin{aligned} x_1 + 2x_2 + 3x_3 + 4x_4 &= 11,\\ 2x_1 + 3x_2 + 4x_3 + x_4 &= 12,\\ 3x_1 + 4x_2 + x_3 + 2x_4 &= 13,\\ 4x_1 + x_2 + 2x_3 + 3x_4 &= 14. \end{aligned}$$

408.

$$\begin{aligned} 2x_1 + 3x_2 - x_3 + 5x_4 &= 0,\\ 3x_1 - x_2 + 2x_3 - 7x_4 &= 0,\\ 4x_1 + x_2 - 3x_3 + 6x_4 &= 0,\\ x_1 - 2x_2 + 4x_3 - 7x_4 &= 0. \end{aligned}$$

409.

$$\begin{aligned} 3x_1 + 4x_2 - 5x_3 + 7x_4 &= 0,\\ 2x_1 - 3x_2 + 3x_3 - 2x_4 &= 0,\\ 4x_1 + 11x_2 - 13x_3 + 16x_4 &= 0,\\ 7x_1 - 2x_2 + x_3 + 3x_4 &= 0. \end{aligned}$$

410. $$\begin{aligned} x_1+x_2-3x_4-x_5&=0,\\ x_1-x_2+2x_3-x_4&=0,\\ 4x_1-2x_2+6x_3+3x_4-4x_5&=0,\\ 2x_1+4x_2-2x_3+4x_4-7x_5&=0. \end{aligned}$$

411. $$\begin{aligned} x_1+x_2+x_3+x_4+x_5&=7,\\ 3x_1+2x_2+x_3+x_4-3x_5&=-2,\\ x_2+2x_3+2x_4+6x_5&=23,\\ 5x_1+4x_2+3x_3+3x_4-x_5&=12. \end{aligned}$$

412. $$\begin{aligned} x_1-2x_2+x_3-x_4+x_5&=0,\\ 2x_1+x_2-x_3+2x_4-3x_5&=0,\\ 3x_1-2x_2-x_3+x_4-2x_5&=0,\\ 2x_1-5x_2+x_3-2x_4+2x_5&=0. \end{aligned}$$

413. $$\begin{aligned} x_1-2x_2+x_3+x_4-x_5&=0,\\ 2x_1+x_2-x_3-x_4+x_5&=0,\\ x_1+7x_2-5x_3-5x_4+5x_5&=0,\\ 3x_1-x_2-2x_3+x_4-x_5&=0. \end{aligned}$$

414. $$\begin{aligned} 2x_1+x_2-x_3-x_4+x_5&=1,\\ x_1-x_2+x_3+x_4-2x_5&=0,\\ 3x_1+3x_2-3x_3-3x_4+4x_5&=2,\\ 4x_1+5x_2-5x_3-5x_4+7x_5&=3. \end{aligned}$$

415. $$\begin{aligned} 2x_1-2x_2+x_3-x_4+x_5&=1,\\ x_1+2x_2-x_3+x_4-2x_5&=1,\\ 4x_1-10x_2+5x_3-5x_4+7x_5&=1,\\ 2x_1-14x_2+7x_3-7x_4+11x_5&=-1. \end{aligned}$$

416. $$\begin{aligned} 3x_1+x_2-2x_3+x_4-x_5&=1,\\ 2x_1-x_2+7x_3-3x_4+5x_5&=2,\\ x_1+3x_2-2x_3+5x_4-7x_5&=3,\\ 3x_1-2x_2+7x_3-5x_4+8x_5&=3. \end{aligned}$$

417. $$\begin{aligned} x_1+2x_2-3x_4+2x_5&=1,\\ x_1-x_2-3x_3+x_4-3x_5&=2,\\ 2x_1-3x_2+4x_3-5x_4+2x_5&=7,\\ 9x_1-9x_2+6x_3-16x_4+2x_5&=25. \end{aligned}$$

418.
$$\begin{aligned} x_1+3x_2+5x_3-4x_4 &= 1,\\ x_1+3x_2+2x_3-2x_4+x_5 &= -1,\\ x_1-2x_2+x_3-x_4-x_5 &= 3,\\ x_1-4x_2+x_3+x_4-x_5 &= 3,\\ x_1+2x_2+x_3-x_4+x_5 &= -1. \end{aligned}$$

419.
$$\begin{aligned} x_1+2x_2+3x_3-x_4 &= 1,\\ 3x_1+2x_2+x_3-x_4 &= 1,\\ 2x_1+3x_2+x_3+x_4 &= 1,\\ 2x_1+2x_2+2x_3-x_4 &= 1,\\ 5x_1+5x_2+2x_3 &= 2. \end{aligned}$$

420.
$$\begin{aligned} x_1-2x_2+3x_3-4x_4+2x_5 &= -2,\\ x_1+2x_2-x_3 -x_5 &= -3,\\ x_1-x_2+2x_3-3x_4 &= 10,\\ x_2-x_3+x_4-2x_5 &= -5,\\ 2x_1+3x_2-x_3+x_4+4x_5 &= 1. \end{aligned}$$

421. The system of equations

$$\begin{aligned} ay+bx &= c,\\ cx+az &= b,\\ bz+cy &= a \end{aligned}$$

has a unique solution. Prove that $abc \neq 0$ and find the solution.

Solve the following systems of equations:

422.
$$\begin{aligned} \lambda x+y+z &= 1,\\ x+\lambda y+z &= \lambda,\\ x+y+\lambda z &= \lambda^2. \end{aligned}$$

423.
$$\begin{aligned} \lambda x+y+z+t &= 1,\\ x+\lambda y+z+t &= \lambda,\\ x+y+\lambda z+t &= \lambda^2,\\ x+y+z+\lambda t &= \lambda^3. \end{aligned}$$

424.
$$\begin{aligned} x+ay+a^2z &= a^3,\\ x+by+b^2z &= b^3,\\ x+cy+c^2z &= c^3. \end{aligned}$$

425.
$$\begin{aligned} x+y+z &= 1,\\ ax+b\,y+c\,z &= d,\\ a^2x+b^2y+c^2z &= d^2. \end{aligned}$$

426.
$$\begin{aligned} ax+y+z &= 4,\\ x+by+z &= 3,\\ x+2by+z &= 4. \end{aligned}$$

427.
$$\begin{aligned} ax+by+z &= 1,\\ x+aby+z &= b,\\ x+by+az &= 1. \end{aligned}$$

428. $$\begin{aligned} \alpha x+y+z&=m,\\ x+\alpha y+z&=n,\\ x+y+\alpha z&=p. \end{aligned}$$

429. $$\begin{aligned} x+ay+a^2z&=1,\\ x+ay+abz&=a,\\ bx+a^2y+a^2bz&=a^2b. \end{aligned}$$

430. $$\begin{aligned} (\lambda+3)x+y+2z&=\lambda,\\ \lambda x+(\lambda-1)y+z&=2\lambda,\\ 3(\lambda+1)x+\lambda y+(\lambda+3)z&=3. \end{aligned}$$

431. $$\begin{aligned} \lambda x+\lambda y+(\lambda+1)z&=\lambda,\\ \lambda x+\lambda y+(\lambda-1)z&=\lambda,\\ (\lambda+1)x+\lambda y+(2\lambda+3)z&=1. \end{aligned}$$

432. $$\begin{aligned} 3kx+(2k+1)y+(k+1)z&=k,\\ (2k-1)x+(2k-1)y+(k-2)z&=k+1,\\ (4k-1)x+3ky+2kz&=1. \end{aligned}$$

433. $$\begin{aligned} ax+by+2z&=1,\\ ax+(2b-1)y+3z&=1,\\ ax+by+(b+3)z&=2b-1. \end{aligned}$$

434. (a) $$\begin{aligned} 3mx+(3m-7)y+(m-5)z&=m-1,\\ (2m-1)x+(4m-1)y+2mz&=m+1,\\ 4mx+(5m-7)y+(2m-5)z&=0. \end{aligned}$$

(b) $$\begin{aligned} (2m+1)x-my+(m+1)z&=m-1,\\ (m-2)x+(m-1)y+(m-2)z&=m,\\ (2m-1)x+(m-1)y+(2m-1)z&=m. \end{aligned}$$

(c) $$\begin{aligned} (5\lambda+1)x+2\lambda y+(4\lambda+1)z&=1+\lambda,\\ (4\lambda-1)x+(\lambda-1)y+(4\lambda-1)z&=-1,\\ 2(3\lambda+1)x+2\lambda y+(5\lambda+2)z&=2-\lambda. \end{aligned}$$

435. (a) $$\begin{aligned} (2c+1)x-cy-(c+1)z&=2c,\\ 3cx-(2c-1)y-(3c-1)z&=c+1,\\ (c+2)x-y-2cz&=2. \end{aligned}$$

(b) $$\begin{aligned} 2(\lambda+1)x+3y+\lambda z&=\lambda+4,\\ (4\lambda-1)x+(\lambda+1)y+(2\lambda-1)z&=2\lambda+2,\\ (5\lambda-4)x+(\lambda+1)y+(3\lambda-4)z&=\lambda-1. \end{aligned}$$

(c) $$\begin{aligned} dx+(2d-1)y+(d+2)z&=1,\\ (d-1)y+(d-3)z&=1+d,\\ dx+(3d-2)y+(3d+1)z&=2-d. \end{aligned}$$

(d) $$\begin{aligned}(3a-1)x+\quad 2ay+(3a+1)z&=1,\\ 2ax+\quad 2ay+(3a+1)z&=a,\\ (a+1)x+(a+1)y+2(a+1)z&=a^2.\end{aligned}$$

436. Find the equation of a straight line passing through the points $M_1(x_1, y_1)$, $M_2(x_2, y_2)$.

437. Under what condition do the three points $M_1(x_1, y_1)$, $M_2(x_2, y_2)$, $M_3(x_3, y_3)$ lie on a straight line?

438. Under what condition do the three straight lines $a_1x+b_1y+c_1=0$, $a_2x+b_2y+c_2=0$, $a_3x+b_3y+c_3=0$ pass through one point?

439. Under what condition do the four points $M_0(x_0, y_0)$, $M_1(x_1, y_1)$, $M_2(x_2, y_2)$, $M_3(x_3, y_3)$ lie on one circle?

440. Write the equation of a circle passing through the points $M_1(2, 1)$, $M_2(1, 2)$, $M_3(0, 1)$.

441. Find the equation of a quadric curve passing through the points $M_1(0, 0)$, $M_2(1, 0)$, $M_3(-1, 0)$, $M_4(1, 1)$ and $M_5(-1, 1)$.

442. Find the equation of a third-degree parabola passing through the points $M_1(1, 0)$, $M_2(0, -1)$, $M_3(-1, -2)$ and $M_4(2, 7)$.

443. Form the equation of a parabola of degree n $y=a_0x^n+a_1x^{n-1}+\ldots+a_n$ passing through the $n+1$ points $M_0(x_0, y_0)$, $M_1(x_1, y_1)$, $M_2(x_2, y_2)$, $\ldots$, $M_n(x_n, y_n)$.

444. Under what condition do the four points $M_1(x_1, y_1, z_1)$, $M_2(x_2, y_2, z_2)$, $M_3(x_3, y_3, z_3)$, $M_4(x_4, y_4, z_4)$ lie in a single plane?

445. Form the equation of a sphere passing through the points $M_1(1, 0, 0)$, $M_2(1, 1, 0)$, $M_3(1, 1, 1)$, $M_4(0, 1, 1)$.

446. Under what condition do the n points $M_1(x_1, y_1)$, $M_2(x_2, y_2)$, $M_3(x_3, y_3)$, $\ldots$, $M_n(x_n, y_n)$ lie on a single straight line?

447. Under what condition are the n straight lines $a_1x+b_1y+c_1=0$, $a_2x+b_2y+c_2=0$, $\ldots$, $a_nx+b_ny+c_n=0$ concurrent?

448. Under what condition do the n points $M_1(x_1, y_1, z_1)$, $M_2(x_2, y_2, z_2)$, $\ldots$, $M_n(x_n, y_n, z_n)$ lie in one plane and under what condition do they lie on one straight line?

449. Under what condition do the n planes $A_ix+B_iy+C_iz+D_i=0$ $(i=1, 2, \ldots, n)$ pass through one point and under what condition do all these planes pass through a single straight line?

450. Eliminate $x_1, x_2, \ldots, x_{n-1}$ from the system of n equations:

$$\begin{aligned}a_{11}x_1+a_{12}x_2+\ldots+a_{1,\,n-1}x_{n-1}+a_{1n}&=0,\\ a_{21}x_1+a_{22}x_2+\ldots+a_{2,\,n-1}x_{n-1}+a_{2n}&=0,\\ \cdots\cdots\cdots\cdots\cdots\cdots&\cdots\\ a_{n1}x_1+a_{n2}x_2+\ldots+a_{n,\,n-1}x_{n-1}+a_{nn}&=0.\end{aligned}$$

451. Let

$$\left.\begin{array}{llll} x_1^{(1)}=\alpha_{11}; & x_2^{(1)}=\alpha_{12}; & \ldots; & x_n^{(1)}=\alpha_{1n}, \\ x_1^{(2)}=\alpha_{21}; & x_2^{(2)}=\alpha_{22}; & \ldots; & x_n^{(2)}=\alpha_{2n}, \\ \ldots & \ldots & \ldots & \ldots \\ x_1^{(m)}=\alpha_{m1}; & x_2^{(m)}=\alpha_{m2}; & \ldots; & x_n^{(m)}=\alpha_{mn} \end{array}\right\} \qquad (1)$$

be m solutions of some system of homogeneous linear equations. These solutions are termed linearly dependent if there exist constants $c_1, c_2, \ldots, c_m$ not all zero, such that

$$c_1\alpha_{1i}+c_2\alpha_{2i}+\ldots+c_m\alpha_{mi}=0 \qquad (2)$$
$$(i=1,\ 2,\ \ldots,\ n).$$

If the equations (2) are only possible when $c_1=c_2=\ \ldots\ =c_m=0$, then the solutions are termed linearly independent.

Let us agree to write the solutions as rows of a matrix.

Thus, the system of solutions of (1) is written in matrix form as

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mn} \end{pmatrix}=A.$$

Prove that if the rank of matrix A is r, then the system (1) has r linearly independent solutions and all other solutions of (1) are linear combinations of them.

452. Prove that if the rank of a system of m homogeneous linear equations in n unknowns is equal to r, then there exist $n-r$ linearly independent solutions of the system, and all other solutions of the system are linear combinations of them.

Such a system of $n-r$ solutions is termed a fundamental system of solutions.

453. Is $\begin{pmatrix} 1 & -2 & 1 & 0 & 0 \\ 1 & -2 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 1 & -2 & 3 & -2 & 0 \end{pmatrix}$ a fundamental system of solutions of the system of equations

$$\begin{aligned} x_1+x_2+x_3+x_4+x_5&=0, \\ 3x_1+2x_2+x_3+x_4-3x_5&=0, \\ x_2+2x_3+2x_4+6x_5&=0, \\ 5x_1+4x_2+3x_3+3x_4-x_5&=0? \end{aligned}$$

454. Write a fundamental system of solutions of the system of equations of Problem 453.

455. Is $\begin{pmatrix} 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 4 & 0 & 0 & -6 & 2 \end{pmatrix}$ a fundamental system of solutions of the system of Problem 453?

456. Prove that if A is a rank r matrix that forms a fundamental system of solutions of a system of homogeneous linear equations, and B is an arbitrary nonsingular matrix of order r, then the matrix BA also forms a fundamental system of solutions of the same system of equations.

457. Prove that if two matrices A and C of rank r form fundamental systems of solutions of some system of homogeneous linear equations, then one of them is the product of some nonsingular matrix B of order r by the other; that is, $A=BC$.

458. Let $\begin{pmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2n} \\ \dots & \dots & \dots & \dots \\ \alpha_{r1} & \alpha_{r2} & \dots & \alpha_{rn} \end{pmatrix}$ be a fundamental system of solutions of some system of homogeneous linear equations. Prove that

$$\begin{aligned} x_1 &= c_1\alpha_{11}+c_2\alpha_{21}+\dots+c_r\alpha_{r1}, \\ x_2 &= c_1\alpha_{12}+c_2\alpha_{22}+\dots+c_r\alpha_{r2}, \\ &\dots\dots\dots\dots\dots\dots \\ x_n &= c_1\alpha_{1n}+c_2\alpha_{2n}+\dots+c_r\alpha_{rn} \end{aligned}$$

is the general solution of this system of equations, i.e., that any solution of the system may be obtained from it for certain values of $c_1, c_2, \dots, c_r$, and conversely.

459. Write the general solution to the system of Problem 453.

460. Verify that $(11 \;\; 1 \; -7)$ is a fundamental system of solutions of the system of Problem 403, and write the general solution.

461. Write the general solutions of the systems of Problems 408, 409, 410, 412, 413.

462. Knowing the general solution of the system of Problem 453 (see the answer to Problem 459) and the fact that $x_1=-16$, $x_2=23$, $x_3=x_4=x_5=0$ is a particular solution of the system of Problem 411, find the general solution of the system of 411.

463. Write the general solutions of the systems of Problems 406, 414, 415.

CHAPTER 4

MATRICES

Sec. 1. Operations on Square Matrices

464. Multiply the matrices:

(a) $\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$, (b) $\begin{pmatrix} 3 & 5 \\ 6 & -1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 1 \\ -3 & 2 \end{pmatrix}$,

(c) $\begin{pmatrix} 3 & 1 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & -1 \\ 2 & -1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$,

(d) $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix} \cdot \begin{pmatrix} -1 & -2 & -4 \\ -1 & -2 & -4 \\ 1 & 2 & 4 \end{pmatrix}$,

(e) $\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 3 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 3 & 1 \\ -1 & 1 & 0 \\ 1 & 2 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 3 & 1 & 1 \end{pmatrix}$,

(f) $\begin{pmatrix} a & b & c \\ c & b & a \\ 1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & a & c \\ 1 & b & b \\ 1 & c & a \end{pmatrix}$.

465. Perform the following operations:

(a) $\begin{pmatrix} 2 & 1 & 1 \\ 3 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix}^2$, (b) $\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}^3$, (c) $\begin{pmatrix} 3 & 2 \\ -4 & -2 \end{pmatrix}^5$,

(d) $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^n$, (e) $\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}^n$.

***466.** Find $\lim\limits_{n\to\infty} \begin{pmatrix} 1 & \dfrac{\alpha}{n} \\ -\dfrac{\alpha}{n} & 1 \end{pmatrix}^n$, where α is a real number.

467. Prove that if $AB=BA$, then

(a) $(A+B)^2=A^2+2AB+B^2$,

(b) $A^2-B^2=(A+B)(A-B)$,

(c) $(A+B)^n=A^n+\frac{n}{1}A^{n-1}B+\ldots+B^n$.

468. Compute $AB-BA$ if:

(a) $A=\begin{pmatrix}1 & 2 & 1\\ 2 & 1 & 2\\ 1 & 2 & 3\end{pmatrix}$, $B=\begin{pmatrix}4 & 1 & 1\\ -4 & 2 & 0\\ 1 & 2 & 1\end{pmatrix}$;

(b) $A=\begin{pmatrix}2 & 1 & 0\\ 1 & 1 & 2\\ -1 & 2 & 1\end{pmatrix}$, $B=\begin{pmatrix}3 & 1 & -2\\ 3 & -2 & 4\\ -3 & 5 & -1\end{pmatrix}$.

469. Find all matrices that commute with the matrix A:

(a) $A=\begin{pmatrix}1 & 2\\ -1 & -1\end{pmatrix}$, (b) $A=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$,

(c) $A=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 3 & 1 & 2\end{pmatrix}$.

470. Find $f(A)$:

(a) $f(x)=x^2-x-1$, $A=\begin{pmatrix}2 & 1 & 1\\ 3 & 1 & 2\\ 1 & -1 & 0\end{pmatrix}$;

(b) $f(x)=x^2-5x+3$, $A=\begin{pmatrix}2 & -1\\ -3 & 3\end{pmatrix}$.

471. Prove that every second-order matrix $A=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ satisfies the equation

$$x^2-(a+d)x+(ad-bc)=0.$$

472. Prove that for any given matrix A there is a polynomial $f(x)$ such that $f(A)=0$, and that all polynomials with this property are divisible by one of them.

***473.** Prove that the equation $AB-BA=E$ is impossible.

474. Let $A^k=0$. Prove that $(E-A)^{-1}=E+A+A^2+\ldots+A^{k-1}$.

475. Find all second-order matrices whose squares are equal to the zero matrix.

476. Find all second-order matrices whose cubes are equal to the zero matrix.

477. Find all second-order matrices whose squares are equal to the unit matrix.

478. Solve and investigate the equation $XA=0$, where A is the given matrix and X is the second-order matrix sought.

479. Solve and investigate the equation $X^2=A$, where A is the given matrix and X is the desired second-order matrix.

480. Find the inverse of the matrix A:

(a) $A=\begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$, (b) $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$,

(c) $A=\begin{pmatrix} 1 & 2 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$, (d) $A=\begin{pmatrix} 1 & 3 & -5 & 7 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$,

(e) $A=\begin{pmatrix} 2 & 2 & 3 \\ 1 & -1 & 0 \\ -1 & 2 & 1 \end{pmatrix}$, (f) $A=\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$,

(g) $A=\begin{pmatrix} 2 & 1 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ 1 & 1 & 3 & 4 \\ 2 & -1 & 2 & 3 \end{pmatrix}$, (h) $A=\begin{pmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ 1 & 1 & 0 & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & 1 & \dots & 0 \end{pmatrix}$,

(i) $A=\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \varepsilon & \varepsilon^2 & \dots & \varepsilon^{n-1} \\ 1 & \varepsilon^2 & \varepsilon^4 & \dots & \varepsilon^{2n-2} \\ \dots & \dots & \dots & \dots & \dots \\ 1 & \varepsilon^{n-1} & \varepsilon^{2n-2} & \dots & \varepsilon^{(n-1)^2} \end{pmatrix}$,

where $\varepsilon=\cos\frac{2\pi}{n}+i\sin\frac{2\pi}{n}$,

(j) $A = \begin{pmatrix} 2 & -1 & 0 & \dots & & 0 \\ -1 & 2 & -1 & \dots & & 0 \\ 0 & -1 & 2 & \dots & & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & -1 & 2 \end{pmatrix}$,

(k) $A = \begin{pmatrix} 1 & 3 & 5 & 7 & \dots & 2n-1 \\ 2n-1 & 1 & 3 & 5 & \dots & 2n-3 \\ 2n-3 & 2n-1 & 1 & 3 & \dots & 2n-5 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 3 & 5 & 7 & 9 & \dots & 1 \end{pmatrix}$,

(l) $A = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & c_1 \\ 0 & 1 & 0 & \dots & 0 & c_2 \\ 0 & 0 & 1 & \dots & 0 & c_3 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & c_n \\ b_1 & b_2 & b_3 & \dots & b_n & a \end{pmatrix}$,

(m) $A = \begin{pmatrix} 1 & -x & 0 & \dots & & 0 \\ 0 & 1 & -x & \dots & & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & -x \\ a_0 & a_1 & a_2 & \dots & & a_n \end{pmatrix}$,

(n) $A = \begin{pmatrix} 1+\frac{1}{\lambda_1} & 1 & 1 & & 1 \\ 1 & 1+\frac{1}{\lambda_2} & 1 & & 1 \\ 1 & 1 & 1+\frac{1}{\lambda_3} & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & 1 & \dots & 1+\frac{1}{\lambda_n} \end{pmatrix}$.

(o) Knowing the matrix B^{-1}, find the inverse of the bordered matrix $\begin{pmatrix} B & U \\ V & a \end{pmatrix}$.

481. Find the desired matrix X from the equations:

(a) $\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix} \cdot X = \begin{pmatrix} 4 & -6 \\ 2 & 1 \end{pmatrix}$,

(b) $X\cdot\begin{pmatrix}1 & 1 & -1\\ 2 & 1 & 0\\ 1 & -1 & 1\end{pmatrix}=\begin{pmatrix}1 & -1 & 3\\ 4 & 3 & 2\\ 1 & -2 & 5\end{pmatrix}$,

(c) $\begin{pmatrix}1 & 1 & 1 & \dots & 1\\ 0 & 1 & 1 & \dots & 1\\ 0 & 0 & 1 & \dots & 1\\ \dots & \dots & \dots & \dots & \dots\\ 0 & 0 & 0 & \dots & 1\end{pmatrix}\cdot X=\begin{pmatrix}2 & 1 & 0 & \dots & 0\\ 1 & 2 & 1 & \dots & 0\\ 0 & 1 & 2 & \dots & 0\\ \dots & \dots & \dots & \dots & \dots\\ 0 & 0 & 0 & \dots & 2\end{pmatrix}$,

(d) $\begin{pmatrix}2 & 1\\ 3 & 2\end{pmatrix}\cdot X\cdot\begin{pmatrix}-3 & 2\\ 5 & -3\end{pmatrix}=\begin{pmatrix}-2 & 4\\ 3 & -1\end{pmatrix}$,

(e) $\begin{pmatrix}1 & 1 & 1 & \dots & 1 & 1\\ -1 & 1 & 0 & \dots & 0 & 0\\ 0 & -1 & 1 & \dots & 0 & 0\\ \dots & \dots & \dots & \dots & \dots & \dots\\ 0 & 0 & 0 & \dots & -1 & 1\end{pmatrix}\cdot X\cdot\begin{pmatrix}1 & -1 & 0 & \dots & 0\\ 1 & 1 & -1 & \dots & 0\\ 1 & 0 & 1 & \dots & 0\\ \dots & \dots & \dots & \dots & \dots\\ 1 & 0 & 0 & \dots & 1\end{pmatrix}$

$$=\begin{pmatrix}1 & 0 & 0 & \dots & 0 & 0\\ 0 & 2 & -1 & \dots & 0 & 0\\ 0 & -1 & 2 & \dots & 0 & 0\\ \dots & \dots & \dots & \dots & \dots & \dots\\ 0 & 0 & 0 & \dots & 2 & -1\\ 0 & 0 & 0 & \dots & -1 & 2\end{pmatrix},$$

(f) $\begin{pmatrix}2 & 1\\ 2 & 1\end{pmatrix}\cdot X=\begin{pmatrix}2 & 1\\ 2 & 1\end{pmatrix}$; (g) $X\cdot\begin{pmatrix}2 & 1\\ 2 & 1\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$.

482. Prove that if $AB=BA$, then $A^{-1}B=BA^{-1}$.

483. Compute $\varphi(A)$, where $\varphi(x)=\frac{1+x}{1-x}$, $A=\begin{pmatrix}1 & 2\\ 2 & 1\end{pmatrix}$.

484. Find all the second-order real matrices whose cubes are equal to the unit matrix.

485. Find all the second-order real matrices whose fourth powers are equal to the unit matrix.

486. Establish that there is an isomorphism between the field of complex numbers and the set of matrices of the form $\begin{pmatrix}a & b\\ -b & a\end{pmatrix}$ for real a, b.

487. Establish that for real a, b, c, d, the matrices of the form $\begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix}$ constitute a ring without zero divisors.

488. Represent $(a_1^2+b_1^2+c_1^2+d_1^2)(a_2^2+b_2^2+c_2^2+d_2^2)$ as a sum of four squares of bilinear expressions.

489. Prove that the following operations involving matrices are accomplished by premultiplication of the matrix by certain nonsingular matrices:

(a) interchanging two rows,

(b) adding, to elements of one row, numbers proportional to the elements of another row,

(c) multiplying elements of a row by a nonzero scalar.

The same operations involving columns are performed via postmultiplication.

490. Prove that every matrix can be represented as PRQ, where P and Q are nonsingular matrices and R is a diagonal matrix of the form

$$R = \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & \ddots & & & \\ & & & 1 & & \\ & & & & 0 & \\ & & & & & \ddots & \\ & & & & & & 0 \end{pmatrix}.$$

***491.** Prove that every matrix may be represented as a product of the matrices $E+\alpha e_{ik}$, where e_{ik} is a matrix whose element of the ith row and kth column is unity, and all other elements are zero.

***492.** Prove that the rank of a product of two square matrices of order n is not less than r_1+r_2-n, where r_1 and r_2 are the ranks of the factors.

493. Prove that every square matrix of rank 1 is of the form

$$\begin{pmatrix} \lambda_1\mu_1 & \lambda_1\mu_2 & \dots & \lambda_1\mu_n \\ \lambda_2\mu_1 & \lambda_2\mu_2 & \dots & \lambda_2\mu_n \\ \dots & \dots & \dots & \dots \\ \lambda_n\mu_1 & \lambda_n\mu_2 & \dots & \lambda_n\mu_n \end{pmatrix}.$$

***494.** Find all third-order matrices whose squares are 0.

***495.** Find all third-order matrices whose squares are equal to the unit matrix.

***496.** Let the rectangular matrices A and B have the same number of rows. By (A, B) denote the matrix obtained by adjoining to A all the columns of B. Prove that the rank of $(A, B) \leqslant$ rank of A + rank of B.

***497.** Prove that if $A^2 = E$, then the rank of $(E+A)$ + the rank of $(E-A) = n$, where n is the order of the matrix A.

***498.** Prove that the matrix A with the property $A^2 = E$ can be represented in the form PBP^{-1}, where P is a nonsingular matrix and B is a diagonal matrix, all elements of which are equal to ± 1.

499. Find the condition which a matrix with integral elements must satisfy so that all the elements of the inverse are also integral.

500. Prove that every nonsingular integral matrix can be represented as PR, where P is an integral unimodular matrix, and R is an integral triangular matrix all the elements of which below the principal diagonal are zero, the diagonal elements are positive, and the elements above the principal diagonal are nonnegative and less than the diagonal elements of that column.

***501.** Combine into a single class all integral matrices which are obtained one from the other by premultiplication by integral unimodular matrices. Compute the number of classes of nth-order matrices with a given determinant k.

502. Prove that every integral matrix can be represented as PRQ, where P and Q are integral unimodular matrices and R is an integral diagonal matrix.

503. Prove that every integral unimodular matrix of second order with determinant 1 can be represented as a product of powers (positive and negative) of the matrices

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

504. Prove that every second-order integral unimodular matrix can be represented in the form of a product of the powers of the matrices

$$A=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} \quad \text{and} \quad C=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}.$$

505. Prove that every third-order integral matrix, different from unit matrix, with positive determinant and satisfying the condition $A^2=E$ can be represented in the form QCQ^{-1}, where Q is an integral unimodular matrix and C is one of the matrices

$$\begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{pmatrix} \quad \text{or} \quad \begin{pmatrix}1 & -1 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{pmatrix}.$$

Sec. 2. Rectangular Matrices. Some Inequalities

506. Multiply the matrices:

(a) $\begin{pmatrix}2 & 1 & 1\\ 3 & 0 & 1\end{pmatrix}$ and $\begin{pmatrix}3 & 1\\ 2 & 1\\ 1 & 0\end{pmatrix}$; (b) $\begin{pmatrix}3 & 2 & 1\\ 0 & 1 & 2\end{pmatrix}$ and $\begin{pmatrix}1\\ 2\\ 3\end{pmatrix}$;

(c) $\begin{pmatrix}2\\ 1\\ 3\end{pmatrix}$ and $(1\ \ 2\ \ 3)$; (d) $(1\ \ 2\ \ 3)$ and $\begin{pmatrix}2\\ 4\\ 1\end{pmatrix}$.

507. Find the determinant of the product of the matrix $\begin{pmatrix}3 & 2 & 1 & 2\\ 4 & 1 & 1 & 3\end{pmatrix}$ by its transpose.

508. Multiply the matrix $\begin{pmatrix}a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\end{pmatrix}$ by its transpose and apply the theorem on the determinant of a product.

509. Express the mth-order minor of the product of two matrices in terms of the minors of the factors.

510. Prove that all the principal (diagonal) minors of the matrix $\bar{A}A$ are nonnegative. Here, A is a real matrix, and $\bar{A}$ is the transpose of A.

511. Prove that if all the principal kth-order minors of the matrix $\bar{A}A$ are zero, then the ranks of the matrices $\bar{A}A$ and A are less than k. Here, A is a real matrix and $\bar{A}$ is its transpose.

512. Prove that the sums of all diagonal minors of a given order k computed for the matrices $\bar{A}A$ and $A\bar{A}$ are the same.

513. Using multiplication of rectangular matrices, prove the identity

$$(a_1^2+a_2^2+\ldots+a_n^2)(b_1^2+b_2^2+\ldots+b_n^2)$$
$$-(a_1 b_1+a_2 b_2+\ldots+a_n b_n)^2=\sum_{i<k}(a_i b_k-a_k b_i)^2 .$$

514. Prove the identity

$$\sum_{i=1}^{n}|a_i|^2\cdot\sum_{i=1}^{n}|b_i|^2-\left|\sum_{i=1}^{n}a_i b_i'\right|^2=\sum_{i<k}|a_i b_k-a_k b_i|^2 .$$

Here, a_i, b_i are complex numbers and b_i' are the conjugates of b_i.

515. Prove the Bunyakovsky inequality

$$\left(\sum_{i=1}^{n}a_i b_i\right)^2\leqslant\sum_{i=1}^{n}a_i^2\cdot\sum_{i=1}^{n}b_i^2$$

for real a_i, b_i by proceeding from the identity of Problem 513.

516. Prove the inequality

$$\left|\sum_{i=1}^{n}a_i b_i'\right|^2\leqslant\sum_{i=1}^{n}|a_i|^2\cdot\sum_{i=1}^{n}|b_i|^2$$

for complex a_i, b_i.

***517.** Let B and C be two real rectangular matrices such that $(B, C)=A$ is a square matrix [the symbol (B, C) has the same meaning as in Problem 496]. Prove that $|A|^2\leqslant|\bar{B}B|\cdot|\bar{C}C|$.

***518.** Let $A=(B, C)$ be a rectangular matrix with real elements. Prove that

$$|\bar{A}A|\leqslant|\bar{B}B|\cdot|\bar{C}C|.$$

519. Let A be the rectangular real matrix

$$A=\begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n}\\ a_{21} & a_{22} & \ldots & a_{2n}\\ \ldots & \ldots & \ldots & \ldots\\ a_{m1} & a_{m2} & \ldots & a_{mn}\end{pmatrix}.$$

Prove that $|A\bar{A}|\leqslant\sum_{k=1}^{n}a_{1k}^2\cdot\sum_{k=1}^{n}a_{2k}^2\ldots\sum_{k=1}^{n}a_{mk}^2$.

520. Let A be a rectangular matrix with complex elements and A^* the transpose of the complex conjugate of A. Prove that the determinant of the matrix A^*A is a nonnegative real number and that this determinant is zero if and only if the rank of A is less than the number of columns.

521. Let $A=(B, C)$ be a complex rectangular matrix. Prove that $|A^*A| \leqslant |B^*B| \cdot |C^*C|$.

522. Prove that if $|a_{ik}| \leqslant M$, then the modulus of the determinant

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

does not exceed $M^n n^{n/2}$.

***523.** Prove that if a_{ik} are real and lie in the interval $0 \leqslant a_{ik} \leqslant$ $\leqslant M$, then the absolute value of the determinant made up of the numbers a_{ik} does not exceed $M^n 2^{-n} \times (n+1)^{\frac{n+1}{2}}$.

524. Prove that for determinants with complex elements the estimate given in Problem 522 is exact and cannot be improved.

525. Prove that for determinants with real elements the estimate given in Problem 522 is exact for $n=2^m$.

526. Prove that the maximum of the absolute value of the determinants of order n having real elements which do not exceed 1 in absolute value is an integer divisible by 2^{n-1}.

***527.** Find the maximum of the absolute value of the determinants of orders 3 and 5 made up of real numbers that do not exceed 1 in absolute value.

***528.** The adjoint of the matrix A is a matrix whose elements are minors of order $n-1$ of the original matrix in the natural order. Prove that the adjoint of the adjoint is equal to the original matrix multiplied by its determinant to the power $n-2$.

***529.** Prove that the mth-order minors of an adjoint matrix are equal to the complementary minors of the appropriate minors of the original matrix multiplied by Δ^{m-1}.

530. Prove that the adjoint of a product of two matrices is equal to the product of the adjoint matrices in that order.

531. Let all combinations of numbers 1, 2, ..., n taken m at a time be labelled in some fashion.

Given an $n\times n$ matrix $A=(a_{ik})$. Let $A_{\alpha\beta}$ be the mth-order minor of A, the row indices of which form a combination with the index α, the column indices, a combination with the index β. Then, using all such minors, we can construct a matrix $A'_m=(A_{\alpha\beta})$ of order C_n^m. In particular, $A'_1=A$, A'_{n-1} is the adjoint of A.

Prove that $(AB)'_m=A'_m B'_m$, $E'_m=E$, $(A^{-1})'_m=(A'_m)^{-1}$.

532. Prove that if A is a "triangular" matrix of the form

$$A=\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{nn} \end{pmatrix}$$

then under an appropriate labelling of the combinations, the matrix A'_m will also be triangular.

533. Prove that the determinant of the matrix A'_m is equal to $|A|^{C_{n-1}^{m-1}}$.

534. Let the pairs (i, k), $i=1, 2, \dots, n$; $k=1, 2, \dots, m$, be labelled in some fashion. The Kronecker product of two square matrices A and B of orders n and m, respectively, is the matrix $C=A\times B$ of order nm with elements $c_{\alpha_1\alpha_2}=a_{i_1 i_2} b_{k_1 k_2}$, where α_1 is the index of the pair (i_1, k_1), α_2 the index of the pair (i_2, k_2). Prove that

(a) $(A_1\pm A_2)\times B=(A_1\times B)\pm(A_2\times B)$,

(b) $A\times(B_1\pm B_2)=(A\times B_1)\pm(A\times B_2)$,

(c) $(A'\times B')\cdot(A''\times B'')=(A'\cdot A'')\times(B'\cdot B'')$.

***535.** Prove that the determinant $A\times B$ is equal to $|A|^m\cdot|B|^n$.

536. Let the matrices A and B of order mn be partitioned into n^2 square submatrices so that they are of the form

$$A=\begin{pmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \dots & \dots & \dots & \dots \\ A_{n1} & A_{n2} & \dots & A_{nn} \end{pmatrix}, \quad B=\begin{pmatrix} B_{11} & B_{12} & \dots & B_{1n} \\ B_{21} & B_{22} & \dots & B_{2n} \\ \dots & \dots & \dots & \dots \\ B_{n1} & B_{n2} & \dots & B_{nn} \end{pmatrix}$$

where A_{ik} and B_{ik} are square matrices of order m. Let their product be C and let it be partitioned in the same way into submatrices C_{ik}. Prove that

$$C_{ik}=A_{i1}B_{1k}+A_{i2}B_{2k}+\dots+A_{in}B_{nk}.$$

Thus, multiplication of submatrices is performed by the same formal rule as when numbers take the place of submatrices.

*537. Let the matrix C of order mn be partitioned into n^2 equal square submatrices. Let the matrices A_{ik} formed from the elements of the separate submatrices commute in pairs under multiplication. Form the "determinant" $\sum \pm A_{1\alpha_1} A_{2\alpha_2} \dots A_{n\alpha_n} = B$ from the matrices A_{ik}. This "determinant" is a certain matrix of order m. Prove that the determinant of the matrix C is equal to the determinant of the matrix B.

CHAPTER 5

POLYNOMIALS AND RATIONAL FUNCTIONS OF ONE VARIABLE

Sec. 1. Operations on Polynomials. Taylor's Formula. Multiple Roots

538. Multiply the polynomials:

(a) $(2x^4-x^3+x^2+x+1)(x^2-3x+1)$,

(b) $(x^3+x^2-x-1)(x^2-2x-1)$.

539. Perform the division (with remainder):

(a) $2x^4-3x^3+4x^2-5x+6$ by x^2-3x+1,

(b) x^3-3x^2-x-1 by $3x^2-2x+1$.

540. Under what condition is the polynomial x^3+px+q divisible by a polynomial of the form x^2+mx-1?

541. Under what condition is the polynomial x^4+px^2+q divisible by a polynomial of the form x^2+mx+1?

542. Simplify the polynomial

$$1-\frac{x}{1}+\frac{x(x-1)}{1\cdot 2}-\ldots+(-1)^n\frac{x(x-1)\ldots(x-n+1)}{n!}.$$

543. Perform the division (with remainder):

(a) $x^4-2x^3+4x^2-6x+8$ by $x-1$,

(b) $2x^5-5x^3-8x$ by $x+3$,

(c) $4x^3+x^2$ by $x+1+i$,

(d) x^3-x^2-x by $x-1+2i$.

544. Using Horner's scheme, compute $f(x_0)$:

(a) $f(x)=x^4-3x^3+6x^2-10x+16$, $x_0=4$,

(b) $f(x)=x^5+(1+2i)x^4-(1+3i)x^2+7$, $x_0=-2-i$.

545. Use the Horner scheme to expand the polynomial $f(x)$ in powers of $x-x_0$:

(a) $f(x)=x^4+2x^3-3x^2-4x+1$, $x_0=-1$;

(b) $f(x)=x^5$, $x_0=1$;

(c) $f(x)=x^4-8x^3+24x^2-50x+90$, $x_0=2$;

(d) $f(x)=x^4+2ix^3-(1+i)x^2-3x+7+i$, $x_0=-i$;

(e) $f(x)=x^4+(3-8i)x^3-(21+18i)x^2$
$-(33-20i)x+7+18i$, $x_0=-1+2i$.

546. Use the Horner scheme to decompose into partial fractions:

(a) $\dfrac{x^3-x+1}{(x-2)^5}$, (b) $\dfrac{x^4-2x^2+3}{(x+1)^5}$.

***547.** Use the Horner scheme to expand in powers of x:

(a) $f(x+3)$ where $f(x)=x^4-x^3+1$,

(b) $(x-2)^4+4(x-2)^3+6(x-2)^2+10(x-2)+20$.

548. Find the values of the polynomial $f(x)$ and its derivatives when $x=x_0$:

(a) $f(x)=x^5-4x^3+6x^2-8x+10$, $x_0=2$,

(b) $f(x)=x^4-3ix^3-4x^2+5ix-1$, $x_0=1+2i$.

549. Give the multiplicity of the root:

(a) 2 for the polynomial $x^5-5x^4+7x^3-2x^2+4x-8$,

(b) -2 for the polynomial $x^5+7x^4+16x^3+8x^2-16x-16$.

550. Determine the coefficient a so that the polynomial x^5-ax^2-ax+1 has -1 for a root of multiplicity not lower than two.

551. Determine A and B so that the trinomial Ax^4+Bx^3+1 is divisible by $(x-1)^2$.

552. Determine A and B so that the trinomial $Ax^{n+1}+Bx^n+1$ is divisible by $(x-1)^2$.

***553.** Prove that the following polynomials have 1 as a triple root:

(a) $x^{2n}-nx^{n+1}+nx^{n-1}-1$,

(b) $x^{2n+1}-(2n+1)x^{n+1}+(2n+1)x^n-1$,

(c) $(n-2m)x^n-nx^{n-m}+nx^m-(n-2m)$.

554. Prove that the polynomial

$$x^{2n+1}-\frac{n(n+1)(2n+1)}{6}x^{n+2}+\frac{(n-1)(n+2)(2n+1)}{2}x^{n+1}$$
$$-\frac{(n-1)(n+2)(2n+1)}{2}x^{n}+\frac{n(n+1)(2n+1)}{6}x^{n-1}-1$$

is divisible by $(x-1)^5$ and is not divisible by $(x-1)^6$.

***555.** Prove that $(x-1)^{k+1}$ divides the polynomial

$$f(x)=a_0x^n+a_1x^{n-1}+\ldots+a_n$$

if and only if

$$\begin{aligned} a_0+a_1+a_2+\ldots+a_n&=0,\\ a_1+2a_2+\ldots+n\,a_n&=0,\\ a_1+4a_2+\ldots+n^2\,a_n&=0,\\ \ldots\ldots\ldots\ldots\ldots\ldots&\ldots\\ a_1+2^k a_2+\ldots+n^k a_n&=0. \end{aligned}$$

556. Determine the multiplicity of the root a of the polynomial

$$\frac{x-a}{2}[f'(x)+f'(a)]-f(x)+f(a)$$

where $f(x)$ is a polynomial.

557. Find the condition under which the polynomial x^5+ax^3+b has a double root different from zero.

558. Find the condition under which the polynomial $x^5+10ax^3+5bx+c$ has a triple root different from zero.

559. Prove that the trinomial $x^n+ax^{n-m}+b$ cannot have nonzero roots above multiplicity two.

560. Find the condition under which the trinomial $x^n+ax^{n-m}+b$ has a nonzero double root.

***561.** Prove that the k-term polynomial

$$a_1x^{p_1}+a_2x^{p_2}+\ldots+a_kx^{p_k}$$

does not have nonzero roots above multiplicity $(k-1)$.

***562.** Prove that every nonzero root of multiplicity $k-1$ of the polynomial

$$a_1x^{p_1}+a_2x^{p_2}+\ldots+a_kx^{p_k}$$

satisfies the equations

$$a_1x^{p_1}\varphi'(p_1)=a_2x^{p_2}\varphi'(p_2)=\ldots=a_kx^{p_k}\varphi'(p_k)$$

where

$$\varphi(t)=(t-p_1)(t-p_2)(t-p_3)\ \dots\ (t-p_k)$$

and conversely.

***563.** Prove that a polynomial is divisible by its derivative if and only if it is equal to $a_0(x-x_0)^n$.

564. Prove that the polynomial

$$1+\frac{x}{1}+\frac{x^2}{1\cdot 2}+\dots+\frac{x^n}{n!}$$

does not have multiple roots.

565. Prove that for x_0 to be a root of multiplicity k of the numerator of the fractional rational function $f(x)=\frac{\varphi(x)}{w(x)}$, the denominator $w(x)$ of which does not vanish for $x=x_0$, it is necessary and sufficient that

$$f(x_0)=f'(x_0)=\dots=f^{(k-1)}(x_0)=0,\quad f^k(x_0)\neq 0.$$

566. Prove that the fractional rational function $f(x)=\frac{\varphi(x)}{w(x)}$ can be represented in the form

$$f(x)=f(x_0)+\frac{f'(x_0)}{1}(x-x_0)+\dots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$
$$+\frac{F(x)}{w(x)}(x-x_0)^{n+1}$$

where $F(x)$ is a polynomial. It is assumed that $w(x_0)\neq 0$ (Taylor's formula for a fractional rational function).

***567.** Prove that if x_0 is a root of multiplicity k of the polynomial $f_1(x)f_2'(x)-f_2(x)f_1'(x)$, then x_0 is a root of multiplicity $k+1$ of the polynomial $f_1(x)f_2(x_0)-f_2(x)f_1(x_0)$ if this latter polynomial is not identically zero, and conversely.

***568.** Prove that if $f(x)$ does not have multiple roots, then $[f'(x)]^2-f(x)f''(x)$ does not have roots of multiplicity higher than $n-1$, where n is the degree of $f(x)$.

***569.** Construct a polynomial $f(x)$ of degree n, for which $[f'(x)]^2-f(x)f''(x)$ has a root x_0 of multiplicity $n-1$, which is not a root of $f(x)$.

Sec. 2. Proof of the Fundamental Theorem of Higher Algebra and Allied Questions

570. Define δ so that for $|x|<\delta$ the polynomial

$$x^5-4x^3+2x$$

is less than 0.1 in absolute value.

571. Define δ so that $|f(x)-f(2)|<0.01$ for all x satisfying the inequality $|x-2|<\delta$; $f(x)=x^4-3x^3+4x+5$.

572. Define M so that for $|x|>M$

$$|x^4-4x^3+4x^2+2|>100.$$

573. Find x so that $|f(x)|<|f(0)|$ where

(a) $f(x)=x^5-3ix^3+4$, (b) $f(x)=x^5-3x^3+4$.

574. Find x so that $|f(x)|<|f(1)|$ where

(a) $f(x)=x^4-4x^3+2$,

(b) $f(x)=x^4-4x^3+6x^2-4x+5$,

(c) $f(x)=x^4-4x+5$.

575. Prove that if $z-i=a(1-i)$, $0<a<\frac{1}{2}$, then

$$|f(z)|<\sqrt{5}$$

where

$$f(z)=(1+i)z^5+(3-5i)z^4-(9+5i)z^3 -7(1-i)z^2+2(1+3i)z+4-i.$$

***576.** Prove that if $f(z)$ is a polynomial different from a constant, then, in arbitrarily small neighbourhood z_0, there is a z_1 such that $|f(z_1)|>|f(z_0)|$.

577. Prove the d'Alembert lemma for a fractional rational function.

578. Prove that the modulus of a fractional rational function reaches its greatest lower bound as the independent variable varies in a closed rectangular domain.

579. It is obvious that the theorem on the existence of a root does not hold for a fractional rational function. Thus, the function $\frac{1}{z}$ has no root. What prevents 'proving' this theorem by the scheme of that for a polynomial?

***580.** Let $f(x)$ be a polynomial or a fractional rational function. Prove that if a is a root of $f(z)-f(a)$ of multiplicity k and $f(a)\neq 0$, then for a sufficiently small ρ there will be, on the circle $|z-a|=\rho$, $2k$ points at which $|f(z)|=|f(a)|$.

***581.** Prove that if a is a root of $f(z)-f(a)$ of multiplicity k, then for a sufficiently small ρ there will be, on the circle $|z-a|=\rho$, $2k$ points at which $\operatorname{Re}\left(f(z)\right)=\operatorname{Re}\left(f(a)\right)$ and $2k$ points at which $\operatorname{Im}\left(f(z)\right)=\operatorname{Im}\left(f(a)\right)$. Here, $f(z)$ is a polynomial or a fractional rational function.

Sec. 3. Factorization into Linear Factors. Factorization into Irreducible Factors in the Field of Reals. Relationships Between Coefficients and Roots

582. Factor the following polynomials into linear factors:

(a) $x^3-6x^2+11x-6$, (b) x^4+4, (c) $x^4+4x^3+4x^2+1$,

(d) x^4-10x^2+1.

***583.** Factor the following polynomials into linear factors:

(a) $\cos(n \arccos x)$,

(b) $(x+\cos\Theta+i\sin\Theta)^n+(x+\cos\Theta-i\sin\Theta)^n$,

(c) $x^m-C_{2m}^2x^{m-1}+C_{2m}^4x^{m-2}-\ldots+(-1)^mC_{2m}^{2m}$.

584. Factor the following polynomials into irreducible real factors:

(a) x^4+4, (b) x^6+27, (c) $x^4+4x^3+4x^2+1$,

(d) $x^{2n}-2x^n+2$, (e) x^4-ax^2+1, $-2<a<2$,

(f) $x^{2n}+x^n+1$.

585. Construct polynomials of lowest degree using the following roots:

(a) double root 1, simple roots 2, 3, and $1+i$,

(b) triple root -1, simple roots 3 and 4,

(c) double root i, simple root $-1-i$.

586. Find a polynomial of lowest degree whose roots are all roots of unity, the degrees of which do not exceed n.

587. Construct a polynomial of lowest degree with real coefficients, using the roots:

(a) double root 1, simple roots 2, 3 and $1+i$,

(b) triple root $2-3i$,

(c) double root i, simple root $-1-i$.

588. Find the greatest common divisor of the polynomials:

(a) $(x-1)^3(x+2)^2(x-3)(x-4)$ and $(x-1)^2(x+2)(x+5)$,

(b) $(x-1)(x^2-1)(x^3-1)(x^4-1)$ and $(x+1)(x^2+1)(x^3+1)\cdot$ $\cdot(x^4+1)$,

(c) $(x^3-1)(x^2-2x+1)$ and $(x^2-1)^3$.

***589.** Find the greatest common divisor of the polynomials

$$x^m-1 \quad \text{and} \quad x^n-1.$$

590. Find the greatest common divisor of the polynomials

$$x^m+a^m \text{ and } x^n+a^n.$$

591. Find the greatest common divisor of the polynomial and its derivative:

(a) $f(x)=(x-1)^3(x+1)^2(x-3)$,

(b) $f(x)=(x-1)(x^2-1)(x^3-1)(x^4-1)$,

(c) $f(x)=x^{m+n}-x^m-x^n+1$.

592. The polynomial $f(x)$ has no multiple roots. Prove that if x_0 is a root of multiplicity $k>1$ of the equation $f\left(\frac{u(x)}{v(x)}\right)=0$, then the equation $f\left(\frac{u'(x)}{v'(x)}\right)=0$ has x_0 as a root of multiplicity $k-1$. It is assumed that $v(x_0)\neq 0$, $v'(x_0)\neq 0$.

593. Prove that x^2+x+1 divides $x^{3m}+x^{3n+1}+x^{3p+2}$.

594. When is $x^{3m}-x^{3n+1}+x^{3p+2}$ divisible by x^2-x+1?

595. What condition is necessary for x^4+x^2+1 to divide $x^{3m}+$ $+x^{3n+1}+x^{3p+2}$?

596. What condition is necessary for x^2+x+1 to divide $x^{2m}+$ $+x^m+1$?

597. Prove that

$$x^{ka_1}+x^{ka_2+1}+\ldots+x^{ka_k+k-1}$$

is divisible by $x^{k-1}+x^{k-2}+\ldots+1$.

598. For what values of m does x^2+x+1 divide $(x+1)^m-x^m-1$?

599. For what values of m does x^2+x+1 divide $(x+1)^m+x^m+1$?

600. For what values of m does $(x^2+x+1)^2$ divide $(x+1)^m-x^m-1$?

601. For what values of m does $(x^2+x+1)^2$ divide $(x+1)^m+x^m+1$?

602. Can $(x^2+x+1)^3$ divide the polynomials $(x+1)^m+x^m+1$ and $(x+1)^m-x^m-1$?

603. Transform the polynomial

$$1-\frac{x}{1}+\frac{x(x-1)}{1\cdot 2}-\ldots+(-1)^n\frac{x(x-1)\ldots(x-n+1)}{1\cdot 2\ldots n}$$

by assigning to x the values 1, 2,..., n in succession. (Compare with Problem 542.)

604. For what values of m does $X_n(x)$ divide $X_n(x^m)$? (X_n is a cyclotomic polynomial.)

Prove the following theorems:

605. If $f(x^n)$ is divisible by $x-1$, then it is also divisible by x^n-1.

606. If $f(x^n)$ is divisible by $(x-a)^k$, then it is also divisible by $(x^n-a^n)^k$ for $a\neq 0$.

607. If $F(x)=f_1(x^3)+xf_2(x^3)$ is divisible by x^2+x+1, then $f_1(x)$ and $f_2(x)$ are divisible by $x-1$.

***068.** If the polynomial $f(x)$ with real coefficients satisfies the inequality $f(x)\geqslant 0$ for all real values of x, then $f(x)=[\varphi_1(x)]^2+[\varphi_2(x)]^2$ where $\varphi_1(x)$ and $\varphi_2(x)$ are polynomials with real coefficients.

609. The polynomial $f(x)=a_0x^n+a_1x^{n-1}+\ldots+a_n$ has the roots $x_1, \ldots, x_n$. What roots do the following polynomials have:

(a) $a_0x^n-a_1x^{n-1}+a_2x^{n-2}-\ldots+(-1)^n a_n$,

(b) $a_nx^n+a_{n-1}x^{n-1}+\ldots+a_0$,

(c) $f(a)+\dfrac{f'(a)}{1}x+\dfrac{f''(a)}{1\cdot 2}x^2+\ldots+\dfrac{f^{(n)}(a)}{n!}x^n$,

(d) $a_0x^n+a_1bx^{n-1}+a_2b^2x^{n-2}+\ldots+a_nb^n$?

610. Find a relationship between the coefficients of the cubic equation $x^3+px^2+qx+r=0$ under which one root is equal to the sum of the other two.

611. Verify that one of the roots of the equation $36x^3-12x^2-5x+1=0$ is equal to the sum of the other two, and solve the equation.

612. Find a relationship among the coefficients of the quartic equation $x^4+ax^3+bx^2+cx+d=0$ under which the sum of two roots is equal to the sum of the other two.

613. Prove that the equation which satisfies the condition of Problem 612 can be reduced to a biquadratic equation by the substitution $x=y+\alpha$ for an appropriate choice of α.

614. Find a relationship among the coefficients of the quartic equation $x^4+ax^3+bx^2+cx+d=0$ under which the product of two roots is equal to the product of the other two.

615. Prove that the equation which satisfies the hypothesis of Problem 614 may be solved by dividing by x^2 and substituting $y=x+\frac{c}{ax}$ (for $a\neq 0$).

616. Using Problems 612 to 615, solve the following equations:

(a) $x^4-4x^3+5x^2-2x-6=0$,

(b) $x^4+2x^3+2x^2+10x+25=0$,

(c) $x^4+2x^3+3x^2+2x-3=0$,

(d) $x^4+x^3-10x^2-2x+4=0$.

617. Define λ so that one of the roots of the equation $x^3-7x+\lambda=0$ is equal to twice the other root.

618. Define a, b, c so that they are roots of the equation

$$x^3-ax^2+bx-c=0.$$

619. Define a, b, c so that they are roots of the equation

$$x^3+ax^2+bx+c=0.$$

620. The sum of two roots of the equation

$$2x^3-x^2-7x+\lambda=0$$

is equal to 1. Determine λ.

621. Determine the relationship between the coefficients of the equation $x^3+px+q=0$ under which $x_3=\frac{1}{x_1}+\frac{1}{x_2}$.

622. Find the sum of the squares of the roots of the polynomial

$$x^n+a_1x^{n-1}+\ldots+a_n.$$

***623.** Solve the equation

$$x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_n = 0$$

knowing the coefficients a_1 and a_2 and that its roots form an arithmetic progression.

624. Do the roots of the equations

(a) $8x^3 - 12x^2 - 2x + 3 = 0,$

(b) $2x^4 + 8x^3 + 7x^2 - 2x - 2 = 0$

form arithmetic progressions?

625. Given the curve

$$y = x^4 + ax^3 + bx^2 + cx + d.$$

Find a straight line whose points M_1, M_2, M_3, M_4 of intersection with the curve intercept three equal segments $M_1M_2 = M_2M_3 = M_3M_4$. Under what condition does this problem have a solution?

***626.** Form a quartic equation whose roots are α, $\frac{1}{\alpha}$, $-\alpha$, $-\frac{1}{\alpha}$.

***627.** Form a sextic equation with the roots

$$\alpha,\ \frac{1}{\alpha},\ 1-\alpha,\ \frac{1}{1-\alpha},\ 1-\frac{1}{\alpha},\ \frac{1}{1-\frac{1}{\alpha}}.$$

628. Let $f(x) = (x - x_1)(x - x_2) \ldots (x - x_n)$.
Find $f'(x_i)$, $f''(x_i)$ and prove that

$$\frac{\partial f'(x_i)}{\partial x_i} = \frac{1}{2} f''(x_i).$$

629. Prove that if $f(x_1) = f''(x_1) = 0$ but $f'(x_1) \neq 0$, then

$$\sum_{i=2}^{n} \frac{1}{x_1 - x_i} = 0.$$

630. The roots of the polynomial $x^n + a_1x^{n-1} + \ldots + a_n$ form an arithmetic progression. Determine $f'(x_i)$.

Sec. 4. Euclid's Algorithm

631. Determine the greatest common divisor of the following polynomials:

(a) $x^4 + x^3 - 3x^2 - 4x - 1$ and $x^3 + x^2 - x - 1$;

(b) $x^5 + x^4 - x^3 - 2x - 1$ and $3x^4 + 2x^3 + x^2 + 2x - 2$;

(c) $x^6-7x^4+8x^3-7x+7$ and $3x^5-7x^3+3x^2-7$;

(d) $x^5-2x^4+x^3+7x^2-12x+10$
and $3x^4-6x^3+5x^2+2x-2$;

(e) $x^6+2x^4-4x^3-3x^2+8x-5$ and x^5+x^2-x+1;

(f) $x^5+3x^4-12x^3-52x^2-52x-12$
and $x^4+3x^3-6x^2-22x-12$;

(g) $x^5+x^4-x^3-3x^2-3x-1$
and $x^4-2x^3-x^2-2x+1$;

(h) x^4-10x^2+1 and $x^4-4\sqrt{2}\,x^3+6x^2+4\sqrt{2}\,x+1$;

(i) $x^4+7x^3+19x^2+23x+10$
and $x^4+7x^3+18x^2+22x+12$;

(j) x^4-4x^3+1 and x^3-3x^2+1;

(k) $2x^6-5x^5-14x^4+36x^3+86x^2+12x-31$
and $2x^5-9x^4+2x^3+37x^2+10x-14$;

(l) $3x^6-x^5-9x^4-14x^3-11x^2-3x-1$ and
$3x^5+8x^4+9x^3+15x^2+10x+9$.

632. Using Euclid's algorithm, choose polynomials $M_1(x)$ and $M_2(x)$ so that $f_1(x)M_2(x)+f_2(x)M_1(x)=\delta(x)$ where $\delta(x)$ is the greatest common divisor of $f_1(x)$ and $f_2(x)$:

(a) $f_1(x)=x^4+2x^3-x^2-4x-2$,
$f_2(x)=x^4+x^3-x^2-2x-2$;

(b) $f_1(x)=x^5+3x^4+x^3+x^2+3x+1$,
$f_2(x)=x^4+2x^3+x+2$;

(c) $f_1(x)=x^6-4x^5+11x^4-27x^3+37x^2-35x+35$,
$f_2(x)=x^5-3x^4+7x^3-20x^2+10x-25$;

(d) $f_1(x)=3x^7+6x^6-3x^5+4x^4+14x^3-6x^2-4x+4$,
$f_2(x)=3x^6-3x^4+7x^3-6x+2$;

(e) $f_1(x)=3x^5+5x^4-16x^3-6x^2-5x-6$,
$f_2(x)=3x^4-4x^3-x^2-x-2$;

(f) $f_1(x)=4x^4-2x^3-16x^2+5x+9$,
$f_2(x)=2x^3-x^2-5x+4$.

633. Using Euclid's algorithm, choose polynomials $M_1(x)$ and $M_2(x)$ so that $f_1(x)\,M_2(x)+f_2(x)\,M_1(x)=1$:

(a) $f_1(x)=3x^3-2x^2+x+2, \quad f_2(x)=x^2-x+1$;

(b) $f_1(x)=x^4-x^3-4x^2+4x+1, \quad f_2(x)=x^2-x-1$;

(c) $f_1(x)=x^5-5x^4-2x^3+12x^2-2x+12,$
$f_2(x)=x^3-5x^2-3x+17$;

(d) $f_1(x)=2x^4+3x^3-3x^2-5x+2,$
$f_2(x)=2x^3+x^2-x-1$;

(e) $f_1(x)=3x^4-5x^3+4x^2-2x+1,$
$f_2(x)=3x^3-2x^2+x-1$;

(f) $f_1(x)=x^5+5x^4+9x^3+7x^2+5x+3,$
$f_2(x)=x^4+2x^3+2x^2+x+1.$

634. Use the method of undetermined coefficients to choose $M_1(x)$ and $M_2(x)$ so that $f_1(x)\,M_2(x)+f_2(x)\,M_1(x)=1$:

(a) $f_1(x)=x^4-4x^3+1, \quad f_2(x)=x^3-3x^2+1$;

(b) $f_1(x)=x^3, \quad f_2(x)=(1-x)^2$;

(c) $f_1(x)=x^4, \quad f_2(x)=(1-x)^4.$

635. Choose polynomials of lowest degree, $M_1(x)$, $M_2(x)$, so that

(a) $(x^4-2x^3-4x^2+6x+1)\,M_1(x)$
$+(x^3-5x-3)\,M_2(x)=x^4$;

(b) $(x^4+2x^3+x+1)\,M_1(x)$
$+(x^4+x^3-2x^2+2x-1)\,M_2(x)=x^3-2x.$

636. Determine the polynomial of lowest degree that yields a remainder of:

(a) $2x$ when divided by $(x-1)^2$ and $3x$ when divided by $(x-2)^3$;

(b) x^2+x+1 when divided by $x^4-2x^3-2x^2+10x-7$ and $2x^2-3$ when divided by $x^4-2x^3-3x^2+13x-10$.

***637.** Find polynomials $M(x)$ and $N(x)$ such that

$$x^m\,M(x)+(1-x)^n\,N(x)=1.$$

638. Let $f_1(x)\,M(x)+f_2(x)\,N(x)=\delta(x)$ where $\delta(x)$ is the greatest common divisor of $f_1(x)$ and $f_2(x)$. What is the greatest common divisor of $M(x)$ and $N(x)$?

639. Separate the multiple factors of the polynomials:

(a) $x^6-6x^4-4x^3+9x^2+12x+4$,

(b) $x^5-10x^3-20x^2-15x-4$,

(c) $x^6-15x^4+8x^3+51x^2-72x+27$,

(d) $x^5-6x^4+16x^3-24x^2+20x-8$,

(e) $x^6-2x^5-x^4-2x^3+5x^2+4x+4$,

(f) $x^7-3x^6+5x^5-7x^4+7x^3-5x^2+3x-1$,

(g) $x^8+2x^7+5x^6+6x^5+8x^4+6x^3+5x^2+2x+1$.

Sec. 5. The Interpolation Problem and Fractional Rational Functions

640. Use Newton's method to construct a polynomial of lowest degree by means of the given table of values:

(a) $\dfrac{x \;|\; 0\;1\;2\;3\;4}{f(x) \;|\; 1\;2\;3\;4\;6}$; (b) $\dfrac{x \;|\; -1\;0\;1\;2\;3}{f(x) \;|\; 6\;5\;0\;3\;2}$;

(c) $\begin{array}{c|cccc} x & 1 & \frac{9}{4} & 4 & \frac{25}{4} \\ \hline f(x) & 1 & \frac{3}{2} & 2 & \frac{5}{2} \end{array}$, find $f(2)$; (d) $\dfrac{x \;|\; 1\;2\;3\;\;4\;\;6}{f(x) \;|\; 5\;6\;1\;-4\;10}$.

641. Use Lagrange's formula to construct a polynomial by the given table of values:

(a) $\dfrac{x \;|\; 1\;2\;3\;4}{y \;|\; 2\;1\;4\;3}$, (b) $\dfrac{x \;|\; 1\;\;i\;\;-1\;\;-i}{y \;|\; 1\;\;2\;\;3\;\;4}$.

***642.** Find $f(x)$ from the following table of values:

$$\frac{x \;|\; 1\;\;\varepsilon_1\;\;\varepsilon_2\;\ldots\;\varepsilon_{n-1}}{f(x) \;|\; 1\;\;2\;\;3\;\ldots\;n}, \quad \varepsilon_k=\cos\frac{2\pi k}{n}+i\sin\frac{2\pi k}{n}.$$

643. A polynomial $f(x)$, whose degree does not exceed $n-1$, takes on values $y_1, y_2, \ldots, y_n$ in the nth roots of unity. Find $f(0)$.

***644.** Prove the following theorem: so that

$$f(x)=\frac{1}{n}[f(x_1)+f(x_2)+\ldots+f(x_n)]$$

for any polynomial $f(x)$ whose degree does not exceed $n-1$ it is necessary and sufficient that the points $x_1, x_2, \ldots, x_n$ be located on a circle with centre at x_0 and that they divide it into equal parts.

***645.** Prove that if the roots $x_1, x_2, \ldots, x_n$ of a polynomial $\varphi(x)$ are all distinct, then

$$\sum_{i=1}^{n} \frac{x_i^s}{\varphi'(x_i)} = 0 \quad \text{for} \quad 0 \leqslant s \leqslant n-2.$$

646. Find the sum $\sum\limits_{i=1}^{n} \frac{x_i^{n-1}}{\varphi'(x_i)}$ (notations are the same as in Problem 645).

647. Derive the Lagrange interpolation formula by solving the following system of equations:

$$\begin{aligned} a_0 + a_1 x_1 + \ldots + a_{n-1} x_1^{n-1} &= y_1, \\ a_0 + a_1 x_2 + \ldots + a_{n-1} x_2^{n-1} &= y_2, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ a_0 + a_1 x_n + \ldots + a_{n-1} x_n^{n-1} &= y_n. \end{aligned}$$

***648.** Use the following table of values to construct a polynomial of lowest degree:

$$\begin{array}{c|ccccc} x & 0 & 1 & 2 & \ldots & n \\ \hline y & 1 & 2 & 4 & \ldots & 2^n \end{array}.$$

***649.** Use the following table of values to construct a polynomial of lowest degree:

$$\begin{array}{c|ccccc} x & 0 & 1 & 2 & \ldots & n \\ \hline y & 1 & a & a^2 & \ldots & a^n \end{array}.$$

***650.** Find a polynomial of degree $2n$ which upon division by $x(x-2)\ldots(x-2n)$ yields a remainder of 1, and upon division by $(x-1)(x-3)\ldots[x-(2n-1)]$ yields a remainder of -1.

***651.** Construct a polynomial of lowest degree, using the table of values

$$\begin{array}{c|cccccc} x & 1 & 2 & 3 & \ldots & n \\ \hline y & 1 & \frac{1}{2} & \frac{1}{3} & \ldots & \frac{1}{n} \end{array}.$$

***652.** Find a polynomial of degree not exceeding $n-1$ that satisfies the condition $f(x) = \frac{1}{x-a}$ at the points $x_1, x_2, \ldots, x_n$, $x_i \neq a$, $i = 1, 2, \ldots, n$.

***653.** Prove that a polynomial of degree $k \leqslant n$ which assumes integral values for $n+1$ successive integral values of the independent variable, takes on integral values for all integral values of the independent variable.

***654.** Prove that a polynomial of degree n, which takes on integral values for $x=0, 1, 4, 9, \ldots, n^2$, assumes integral values for all squares of the natural numbers.

***655.** Decompose into partial fractions of the first type:

(a) $\frac{x^2}{(x-1)(x+2)(x+3)}$, (b) $\frac{1}{(x-1)(x-2)(x-3)(x-4)}$,

(c) $\frac{3+x}{(x-1)(x^2+1)}$, (d) $\frac{x^2}{x^4-1}$, (e) $\frac{1}{x^3-1}$,

(f) $\frac{1}{x^4+4}$, (g) $\frac{1}{x^n-1}$, (h) $\frac{1}{x^n+1}$,

(i) $\frac{n!}{x(x-1)(x-2)\ldots(x-n)}$,

(j) $\frac{(2n)!}{x(x^2-1)(x^2-4)\ldots(x^2-n^2)}$, (k) $\frac{1}{\cos(n \arccos x)}$.

***656.** Decompose into real partial fractions of the first and second types:

(a) $\frac{1}{x^3-1}$; (b) $\frac{x^2}{x^4-16}$; (c) $\frac{1}{x^4+4}$; (d) $\frac{x^2}{x^6+27}$;

(e) $\frac{x^m}{x^{2n+1}-1}$, $m<2n+1$;

(f) $\frac{x^m}{x^{2n+1}+1}$, $m<2n+1$;

(g) $\frac{1}{x^{2n}-1}$; (h) $\frac{x^{2m}}{x^{2n}+1}$, $m<n$;

(i) $\frac{1}{x(x^2+1)(x^2+4)\ldots(x^2+n^2)}$.

***657.** Decompose into partial fractions of the first type:

(a) $\frac{x}{(x^2-1)^2}$; (b) $\frac{1}{(x^2-1)^2}$; (c) $\frac{5x^2+6x-23}{(x-1)^3(x+1)^2(x-2)}$;

(d) $\frac{1}{(x^n-1)^2}$; (e) $\frac{1}{x^m(1-x)^n}$; (f) $\frac{1}{(x^2-a^2)^n}$, $a \neq 0$;

(g) $\frac{1}{(x^2+a^2)^n}$; (h) $\frac{g(x)}{[f(x)]^2}$

where $f(x)=(x-x_1)(x-x_2)\ldots(x-x_n)$ is a polynomial with no multiple roots and $g(x)$ is a polynomial of degree less than $2n$.

658. Decompose into real partial fractions of the first and second type:

(a) $\frac{x}{(x+1)(x^2+1)^2}$, (b) $\frac{2x-1}{x(x+1)^2(x^2+x+1)^2}$,

(c) $\frac{1}{(x^4-1)^2}$, (d) $\frac{1}{(x^{2n}-1)^2}$.

659. Let $\varphi(x)=(x-x_1)(x-x_2)\dots(x-x_n)$.
Express the following sums in terms of $\varphi(x)$:

(a) $\sum \frac{1}{x-x_i}$; (b) $\sum \frac{x_i}{x-x_i}$; (c) $\sum \frac{1}{(x-x_i)^2}$.

***660.** Compute the following sums, knowing that $x_1, x_2, \dots$ are roots of the polynomial $\varphi(x)$:

(a) $\frac{1}{2-x_1}+\frac{1}{2-x_2}+\frac{1}{2-x_3}$, $\varphi(x)=x^3-3x-1$;

(b) $\frac{1}{x_1^2-3x_1+2}+\frac{1}{x_2^2-3x_2+2}+\frac{1}{x_3^2-3x_3+2}$,

$\varphi(x)=x^3+x^2-4x+1$;

(c) $\frac{1}{x_1^2-2x_1+1}+\frac{1}{x_2^2-2x_2+1}+\frac{1}{x_3^2-2x_3+1}$,

$\varphi(x)=x^3+x^2-1$

661. Determine the first-degree polynomial which approximately assumes the following table of values:

x	0	1	2	3	4
y	2.1	2.5	3.0	3.6	4.1

so that the sum of the squares of the errors is a minimum.

662. Determine the second-degree polynomial which approximately assumes the table of values

x	0	1	2	3	4
y	1	1.4	2	2.7	3.6

so that the sum of the squares of the errors is a minimum.

Sec. 6. Rational Roots of Polynomials. Reducibility and Irreducibility over the Field of Rationals

663. Prove that if $\frac{p}{q}$ is a simplified rational fraction that is a root of the polynomial $f(x)=a_0x^n+a_1x^{n-1}+\dots+a_n$ with integral coefficients, then

(1) q is a divisor of a_0,

(2) p is a divisor of a_n,

(3) $p-mq$ is a divisor of $f(m)$ for any integral m. In particular, $p-q$ is a divisor of $f(1)$, $p+q$ is a divisor of $f(-1)$.

664. Find the rational roots of the following polynomials:

(a) $x^3-6x^2+15x-14$,

(b) $x^4-2x^3-8x^2+13x-24$,

(c) $x^5-7x^3-12x^2+6x+36$,

(d) $6x^4+19x^3-7x^2-26x+12$,

(e) $24x^4-42x^3-77x^2+56x+60$,

(f) $x^5-2x^4-4x^3+4x^2-5x+6$,

(g) $24x^5+10x^4-x^3-19x^2-5x+6$,

(h) $10x^4-13x^3+15x^2-18x-24$,

(i) $x^4+2x^3-13x^2-38x-24$,

(j) $2x^3+3x^2+6x-4$, (k) $4x^4-7x^2-5x-1$,

(l) $x^4+4x^3-2x^2-12x+9$,

(m) $x^5+x^4-6x^3-14x^2-11x-3$,

(n) $x^6-6x^5+11x^4-x^3-18x^2+20x-8$.

***665.** Prove that a polynomial $f(x)$ with integral coefficients has no integral roots if $f(0)$ and $f(1)$ are odd numbers.

***666.** Prove that if a polynomial with integral coefficients assumes the values ± 1 for two integral values x_1 and x_2 of the independent variable, then it has no rational roots if $|x_1-x_2|>2$. However, if $|x_1-x_2|\leqslant 2$, then the only possible rational root is $\frac{1}{2}(x_1+x_2)$.

***667.** Use the Eisenstein criterion to prove the irreducibility of the polynomials

(a) $x^4-8x^3+12x^2-6x+2$,

(b) $x^5-12x^3+36x-12$, (c) x^4-x^3+2x+1.

***668.** Prove the irreducibility of the polynomial

$$X_p(x)=\frac{x^p-1}{x-1}, \quad p \text{ prime.}$$

***669.** Prove the irreducibility of the polynomial

$$X_{p^k}(x)=\frac{x^{p^k}-1}{x^{p^{k-1}}-1}, \quad p \text{ prime.}$$

***670.** Prove that the polynomial $f(x)=a_0x^n+a_1x^{n-1}+\ldots+a_n$ with integral coefficients that has no rational roots is irreducible if there exists a prime number p such that a_0 is not divisible by p; $a_2, a_3, \ldots, a_n$ are divisible by p and a_n is not divisible by p^2.

***671.** Let $f(x)$ be a polynomial with integral coefficients for which there is a prime number p such that a_0 is not divisible by p; $a_{k+1}, a_{k+2}, \ldots, a_n$ are divisible by p and a_n is not divisible by p^2. Prove that in that case $f(x)$ has an irreducible factor of degree $\geqslant n-k$.

672. Using the method of factorization into factors of the values of a polynomial with integral values of the variable, decompose the following polynomials into factors or prove their irreducibility:

(a) x^4-3x^2+1, (b) $x^4+5x^3-3x^2-5x+1$,

(c) $x^4+3x^3-2x^2-2x+1$,

(d) $x^4-x^3-3x^2+2x+2$.

673. Prove that a polynomial of degree three is irreducible if it has no rational roots.

674. Prove that the fourth-degree polynomial $x^4+ax^3+bx^2+$ $+cx+d$ with integral coefficients is irreducible if it has no integral roots and is not divisible by any polynomial of the form

$$x^2+\frac{cm-am^3}{d-m^2}\,x+m$$

where m are divisors of the number d. Polynomials with fractional coefficients may be disregarded. Polynomials like those of Problems 614, 615 are a possible exception.

675. Prove that the fifth-degree polynomial $x^5+ax^4+bx^3+$ $+cx^2+dx+e$ with integral coefficients is irreducible if it has no integral roots and is not divisible by any polynomials with integral coefficients of the following form:

$$x^2+\frac{am^3-cm^2-dn+be}{m^3-n^2+ae-dm}\,x+m$$

where m is a divisor of e, $n=\dfrac{e}{m}$.

676. Factor the following polynomials and prove their irreducibility using Problems 674, 675:

(a) $x^4-3x^3+2x^2+3x-9$, (b) $x^4-3x^3+2x^2+2x-6$,

(c) $x^4+4x^3-6x^2-23x-12$,

(d) $x^5+x^4-4x^3+9x^2-6x+6$.

677. Find the necessary and sufficient conditions for reducibility of the polynomial x^4+px^2+q with rational (possibly fractional) coefficients.

678. Prove that, for the reducibility of a fourth-degree polynomial without rational roots, it is necessary (but not sufficient) that there exists a rational root of a cubic equation obtained in a solution by the Ferrari method.

***679.** Prove the irreducibility of the polynomial $f(x)=(x-a_1)(x-a_2)\dots(x-a_n)-1$; $a_1, a_2, \dots, a_n$ are distinct integers.

***680.** Prove the irreducibility of the polynomial $f(x)=(x-a_1)(x-a_2)\dots(x-a_n)+1$ for distinct integers $a_1, a_2, \dots, a_n$ with the exception of

$$(x-a)(x-a-1)(x-a-2)(x-a-3)+1 = [(x-a-1)(x-a-2)-1]^2$$

and

$$(x-a)(x-a-2)+1=(x-a-1)^2.$$

***681.** Prove that if an nth-degree polynomial with integral coefficients assumes the values ± 1 for more than $2m$ integral values of the variable ($n=2m$ or $2m+1$), then it is irreducible.

***682.** Prove the irreducibility of the polynomial

$$f(x)=(x-a_1)^2(x-a_2)^2\dots(x-a_n)^2+1$$

if $a_1, a_2, \dots, a_n$ are distinct integers.

***683.** Prove that the polynomial $f(x)$ with integral coefficients which takes on the value $+1$ for more than three integral values of the independent variable cannot assume the value -1 for integral values of the independent variable.

***684.** Prove that an nth-degree polynomial with integral coefficients assuming the values ± 1 for more than $\frac{n}{2}$ integral values of the independent variable is irreducible for $n \geqslant 12$.

***685.** Prove that if a polynomial ax^2+bx+1 with integral coefficients is irreducible, then so is the polynomial $a[\varphi(x)]^2+b\varphi(x)+1$ where $\varphi(x)=(x-a_1)(x-a_2)\dots(x-a_n)$ for $n\geqslant 7$. Here, $a_1, a_2, \dots, a_n$ are distinct integers.

Sec. 7. Bounds of the Roots of a Polynomial

686. Prove that the roots of the polynomial $a_0x^n+a_1x^{n-1}+ + \ldots +a_n$ with real or complex coefficients do not exceed, in absolute value,

(a) $1+\max\left|\frac{a_k}{a_0}\right|$, $k=1, 2, \ldots, n$;

(b) $\rho+\max\left|\frac{a_k}{a_0\rho^{k-1}}\right|$,

$k=1, 2, \ldots, n$; ρ is any positive number;

(c) $2\max\sqrt[k]{\left|\frac{a_k}{a_0}\right|}$, $k=1, 2, \ldots, n$;

(d) $\left|\frac{a_1}{a_0}\right|+\max\sqrt[k-1]{\left|\frac{a_k}{a_1}\right|}$, $k=1, 2, \ldots, n$.

687. Prove that the moduli of the roots of the polynomial $a_0x^n+a_1x^{n-1}+ \ldots +a_n$ do not exceed a unique positive root of the equation $b_0x^n-b_1x^{n-1}-b_2x^{n-2}- \ldots -b_n$, where $0< <b_0\leqslant|a_0|$, $b_1\geqslant|a_1|$, $b_2\geqslant|a_2|$, $\ldots$, $b_n\geqslant|a_n|$.

688. Prove that the moduli of the roots of the polynomial $f(x)=a_0x^n+a_r x^{n-r}+ \ldots +a_n$, $a_r\neq 0$, do not exceed

(a) $1+\sqrt[r]{\max\left|\frac{a_k}{a_0}\right|}$, $k=r, \ldots, n$;

(b) $\rho+\sqrt[r]{\max\left|\frac{a_k}{a_0\rho^{k-r}}\right|}$,

$k=r, \ldots, n$, and ρ is any positive number;

(c) $\sqrt[r]{\left|\frac{a_r}{a_0}\right|}+\max\sqrt[k-r]{\left|\frac{a_k}{a_r}\right|}$, $k=r, \ldots, n$.

689. Prove that the real roots of a polynomial with real coefficients do not exceed a unique nonnegative root of a polynomial obtainable from the given polynomial by deleting all terms (except the highest-degree one), the coefficients of which are of sign that coincides with the sign of the leading coefficient.

Prove the following theorems:

690. The real roots of the polynomial $a_0x^n+a_1x^{n-1}+\ldots+a_n$ with real coefficients (for $a_0>0$) do not exceed

(a) $1+\sqrt[r]{\max\left|\frac{a_k}{a_0}\right|}$ where r is the index of the first negative coefficient and a_k are negative coefficients of the polynomial;

(b) $\rho+\sqrt[r]{\max\left|\frac{a_k}{a_0\,\rho^{k-r}}\right|}$ where r is the index of the first negative coefficient, a_k are negative coefficients, and ρ is any positive number;

(c) $2\max\sqrt[k]{\frac{|a_k|}{a_0}}$, a_k are negative coefficients of the polynomial;

(d) $\sqrt[r]{\frac{|a_r|}{a_0}}+\max\sqrt[k-r]{\left|\frac{a_k}{a_r}\right|}$, r is the index of the first negative coefficient, and a_k are negative coefficients.

691. If all the coefficients of the polynomial $f(x)$ are nonnegative, then the polynomial does not have positive roots.

692. If $f(a)>0$, $f'(a)\geqslant 0$, $\ldots$, $f^{(n)}(a)\geqslant 0$, then all real roots of the polynomial do not exceed a.

693. Indicate the upper and lower bounds of the real roots of the polynomials:

(a) $x^4-4x^3+7x^2-8x+3$, (b) x^5+7x^3-3,

(c) $x^7-108x^5-445x^3+900x^2+801$,

(d) $x^4+4x^3-8x^2-10x+14$.

Sec. 8. Sturm's Theorem

694. Form the Sturm polynomials and isolate the roots of the polynomials:

(a) x^3-3x-1, (b) x^3+x^2-2x-1,

(c) x^3-7x+7, (d) x^3-x+5,

(e) x^3+3x-5.

695. Form the Sturm polynomials and isolate the roots of the polynomials:

(a) $x^4-12x^2-16x-4$, (b) x^4-x-1,

(c) $2x^4-8x^3+8x^2-1$, (d) x^4+x^2-1,

(e) $x^4+4x^3-12x+9$.

696. Form the Sturm polynomials and isolate the roots of the following polynomials:

(a) $x^4-2x^3-4x^2+5x+5$, (b) $x^4-2x^3+x^2-2x+1$,

(c) $x^4-2x^3-3x^2+2x+1$, (d) $x^4-x^3+x^2-x-1$,

(e) $x^4-4x^3-4x^2+4x+1$.

697. Form a Sturm sequence and isolate the roots of the following polynomials:

(a) $x^4-2x^3-7x^2+8x+1$, (b) x^4-4x^2+x+1,

(c) $x^4-x^3-x^2-x+1$, (d) $x^4-4x^3+8x^2-12x+8$,

(e) x^4-x^3-2x+1.

698. Form a Sturm sequence and isolate the roots of the following polynomials:

(a) x^4-6x^2-4x+2, (b) $4x^4-12x^2+8x-1$,

(c) $3x^4+12x^3+9x^2-1$, (d) $x^4-x^3-4x^2+4x+1$,

(e) $9x^4-126x^2-252x-140$.

699. Form a Sturm sequence and isolate the roots of the following polynomials:

(a) $2x^5-10x^3+10x-3$,

(b) $x^6-3x^5-3x^4+11x^3-3x^2-3x+1$,

(c) $x^5+x^4-4x^3-3x^2+3x+1$, (d) $x^5-5x^3-10x^2+2$.

700. Form a Sturm sequence using the permission to divide the Sturm functions by positive quantities, and isolate the roots of the following polynomials:

(a) x^4+4x^2-1, (b) $x^4-2x^3+3x^2-9x+1$,

(c) $x^4-2x^3+2x^2-6x+1$,

(d) $x^5+5x^4+10x^2-5x-3$.

701. Use Sturm's theorem to determine the number of real roots of the equation $x^3+px+q=0$ for p and q real.

***702.** Determine the number of real roots of the equation

$$x^n+px+q=0.$$

703. Determine the number of real roots of the equation

$$x^5-5ax^3+5a^2x+2b=0.$$

704. Prove that if a Sturm sequence contains polynomials of all degrees from zero to n, then the number of variations of sign in the sequence of leading coefficients of Sturm polynomials is equal to the number of pairs of conjugate complex roots of the original polynomial.

705. Prove that if the polynomials $f(x)$, $f_1(x)$, $f_2(x)$, ..., $f_k(x)$ have the following properties:

(1) $f(x)f_1(x)$ changes sign from plus to minus when passing through the root $f(x)$;

(2) two adjacent polynomials do not vanish simultaneously;

(3) if $f_\lambda(x_0)=0$, then $f_{\lambda-1}(x_0)$ and $f_{\lambda+1}(x_0)$ have opposite signs;

(4) the last polynomial $f_k(x)$ does not change sign in the interval (a, b),

then the number of roots of the polynomial $f(x)$ in the interval (a, b) is equal to the increment in the number of variations of sign in the sequence of values of the polynomials $f, f_1, \ldots, f_k$ when going from a to b.

706. Let x_0 be a real root of $f'(x)$:

$$f_1(x)=\frac{1}{x-x_0}f'(x);$$

$f_2(x)$ is the remainder, after division of $f(x)$ by $f_1(x)$, taken with reversed sign; $f_3(x)$ is the remainder, left after dividing $f_1(x)$ by $f_2(x)$, taken with reversed sign, and so on. It is assumed that $f(x)$ has no multiple roots. Relate the number of real roots of $f(x)$ to the number of variations in sign in the sequence of values of the polynomials constructed for $x=-\infty$, $x=x_0$, and $x=+\infty$.

***707.** Construct a Sturm sequence for the Hermite polynomials

$$P_n(x)=(-1)^n e^{\frac{x^2}{2}} \frac{d^n e^{-\frac{x^2}{2}}}{dx^n}$$

and determine the number of real roots.

***708.** Determine the number of real roots of the Laguerre polynomials

$$P_n(x)=(-1)^n e^x \frac{d^n(e^{-x}x^n)}{dx^n}.$$

Determine the number of real roots of the following polynomials:

*709. $E_n(x) = 1 + \frac{x}{1} + \frac{x^2}{1 \cdot 2} + \ldots + \frac{x^n}{n!}$.

*710. $P_n(x) = (-1)^{n+1} x^{2n+2} e^{-\frac{1}{x}} \frac{d^{n+1}\left(e^{\frac{1}{x}}\right)}{dx^{n+1}}$.

*711. $P_n(x) = \frac{(-1)^n}{n!} (x^2+1)^{n+1} \frac{d^n}{dx^n}\left(\frac{1}{x^2+1}\right)$.

*712. $P_n(x) = (-1)^n (x^2+1)^{n+\frac{1}{2}} \frac{d^n}{dx^n}\left(\frac{1}{\sqrt{x^2+1}}\right)$.

*713. Let $f(x)$ be a third-degree polynomial without multiple roots. Show that the polynomial $F(x) = 2f(x)f''(x) - [f'(x)]^2$ has two and only two real roots. Investigate the case when $f(x)$ has a double or a triple root.

714. Prove that if all roots of the polynomial $f(x)$ are real and distinct, then all roots of each of the polynomials of the Sturm sequence constructed by the Euclidean algorithm are real and distinct.

Sec. 9. Theorems on the Distribution of Roots of a Polynomial

Prove the following theorems:

715. All roots of the Legendre polynomial $P_n(x) = \frac{d^n (x^2-1)^n}{dx^n}$ are real, distinct and are in the interval $(-1, +1)$.

716. If all the roots of the polynomial $f(x)$ are real, then all the roots of the polynomial $\lambda f(x) + f'(x)$ are real for arbitrary real λ.

*717. If all the roots of the polynomial $f(x)$ are real and all the roots of the polynomial $g(x) = a_0x^n + a_1x^{n-1} + \ldots + a_n$ are real, then all the roots of the polynomial

$$F(x) = a_0 f(x) + a_1 f'(x) + \ldots + a_n f^{(n)}(x)$$

are real.

*718. If all the roots of the polynomial $f(x) = a_0x^n + a_1x^{n-1} + \ldots + a_n$ are real, then all the roots of the polynomial

$$a_0 x^n + a_1 m x^{n-1} + a_2 m(m-1) x^{n-2} + \ldots + a_n m(m-1) \ldots (m-n+1)$$

are real for arbitrary positive integral m.

***719.** If all the roots of the polynomial $f(x)=a_0x^n+a_1x^{n-1}+$ $+\ \ldots\ +a_n$ are real, then all the roots of the polynomial

$$G(x)=a_0x^n+C_n^1a_1x^{n-1}+C_n^2a_2x^{n-2}+\ldots+a_n$$

are real.

720. Prove that all the roots of the following polynomial are real:

$$x^n+\left(\frac{n}{1}\right)^2x^{n-1}+\left(\frac{n(n-1)}{1\cdot 2}\right)^2x^{n-2}+\ldots+1.$$

***721.** Determine the number of real roots of the polynomial

$$nx^n-x^{n-1}-x^{n-2}-\ldots-1.$$

722. Determine the number of real roots of the polynomial

$$x^{2n_1+1}+x^{2n_2+1}+\ldots+x^{2n_k+1}+a.$$

723. Determine the number of real roots of the polynomial $f(x)=(x-a)(x-b)(x-c)-A^2(x-a)-B^2(x-b)-C^2(x-c)$ for a, b, c, A, B, C.

724. Prove that

$$\varphi(x)=\frac{A_1^2}{x-a_1}+\frac{A_2^2}{x-a_2}+\ldots+\frac{A_n^2}{x-a_n}+B$$

does not have imaginary roots for real $a_1, a_2, \ldots, a_n, A_1, A_2, \ldots, A_n, B$.

Prove the following theorems:

725. If the polynomial $f(x)$ has real and distinct roots, then $[f'(x)]^2-f(x)f''(x)$ does not have real roots.

726. If the roots of the polynomials $f(x)$ and $\varphi(x)$ are all real, prime and can be separated, that is, between any two roots of $f(x)$ there is a root of $\varphi(x)$ and between any two roots of $\varphi(x)$ there is a root of $f(x)$, then all roots of the equation $\lambda f(x)+\mu\varphi(x)=$ $=0$ are real for arbitrary real λ and μ.

***727.** If all the roots of the polynomials $F(x)=\lambda f(x)+\mu\varphi(x)$ are real for arbitrary real λ and μ, then the roots of the polynomials $f(x)$ and $\varphi(x)$ can be separated.

***728.** If all the roots of $f'(x)$ are real and distinct and $f(x)$ does not have multiple roots, then the number of real roots of the polynomial $[f'(x)]^2-f(x)f''(x)$ is equal to the number of imaginary roots of the polynomial $f(x)$.

***729.** If the roots of the polynomials $f_1(x)$ and $f_2(x)$ are all real and separable, then the roots of their derivatives can be separated.

***730.** If all the roots of the polynomial $f(x)$ are real, then all the roots of the polynomial $F(x)=\gamma f(x)+(\lambda+x)f'(x)$ are real for $\gamma>0$ or $\gamma<-n$ and for arbitrary real λ as well.

***731.** If the polynomial

$$f(x)=a_0+a_1x+\ldots+a_nx^n$$

has only real roots, and the polynomial

$$\varphi(x)=b_0+b_1x+\ldots+b_kx^k$$

has real roots that do not lie in the interval $(0, n)$, then all the roots of the polynomial

$$a_0\varphi(0)+a_1\varphi(1)x+a_2\varphi(2)x^2+\ldots+a_n\varphi(n)x^n$$

are real.

***732.** If all the roots of the polynomial $f(x)=a_0+a_1x+\ldots+a_nx^n$ are real, then all the roots of the polynomial $a_0+a_1\gamma x+a_2\gamma(\gamma-1)x^2+\ldots+a_n\gamma(\gamma-1)\ldots(\gamma-n+1)x^n$ are real for $\gamma>n-1$.

***733.** If all the roots of the polynomial $f(x)=a_0+a_1x+\ldots+a_nx^n$ are real, then also real are the roots of the polynomial

$$a_0+\frac{\gamma}{\alpha}a_1x+\frac{\gamma(\gamma-1)}{\alpha(\alpha+1)}a_2x^2+\ldots+\frac{\gamma(\gamma-1)\ldots(\gamma-n+1)}{\alpha(\alpha+1)\ldots(\alpha+n-1)}a_nx^n$$

for $\gamma>n-1$, $\alpha>0$.

***734.** If all the roots of the polynomial $f(x)=a_0+a_1x+\ldots+a_nx^n$ are real, then all the roots of the polynomial

$$a_0+a_1wx+a_2w^4x^2+\ldots+a_nw^{n^2}x^n$$

are real for $0<w\leqslant 1$.

***735.** If all the roots of the polynomial $a_0+a_1x+a_2x^2+\ldots+a_nx^n$ are real and of the same sign, then all the roots of the polynomial $a_0\cos\varphi+a_1\cos(\varphi+\theta)x+a_2\cos(\varphi+2\theta)x^2+\ldots+a_n\cos(\varphi+n\theta)x^n$ are real.

***736.** If all the roots of the polynomial

$$(a_0+ib_0)+(a_1+ib_1)x+\ldots+(a_n+ib_n)x^n$$

lie in the upper half-plane, then all the roots of the polynomial

$$a_0+a_1x+\ldots+a_nx^n \text{ and } b_0+b_1x+\ldots+b_nx^n$$

are real and separable (the numbers $a_0, a_1, \ldots, a_n, b_0, b_1, \ldots, b_n$ are real).

***737.** If all the roots of the polynomials $\varphi(x)$ and $\psi(x)$ are real and separable, then the imaginary parts of the roots $\varphi(x)+i\psi(x)$ have the same signs.

***738.** If all the roots of the polynomial $f(x)$ lie in the upper half-plane, then all the roots of its derivative likewise lie in the upper half-plane.

***739.** If all the roots of the polynomial $f(x)$ are located in some half-plane, then all the roots of the derivative are located in the same half-plane.

***740.** The roots of the derivative of the polynomial $f(x)$ lie within an arbitrary convex contour which contains all the roots of the polynomial $f(x)$.

***741.** If $f(x)$ is a polynomial of degree n with real roots, then all the roots of the equation $[f(x)]^2+k^2+[f'(x)]^2=0$ have an imaginary part less than kn in absolute value.

742. If all the roots of the polynomials $f(x)-a$ and $f(x)-b$ are real, then all the roots of the polynomial $f(x)-\lambda$ are real if λ lies between a and b.

***743.** For the real parts of all the roots of the polynomial $x^n+ +a_1x^{n-1}+\ldots+a_n$ with real coefficients to be of the same sign, it is necessary and sufficient that the roots of the polynomials

$$x^n-a_2x^{n-2}+a_4x^{n-4}-\ldots$$

and

$$a_1x^{n-1}-a_3x^{n-3}+\ldots$$

be real and separable.

***744.** Find the necessary and sufficient conditions for the real parts of all the roots of the equation $x^3+ax^2+bx+c=0$ with real coefficients to be negative.

***745.** Find the necessary and sufficient conditions for the negativity of the real parts of all the roots of the equation $x^4+ax^3+ +bx^2+cx+d=0$ with real coefficients.

***746.** Find the necessary and sufficient conditions for all the roots of the equation $x^3+ax^2+bx+c=0$ with real coefficients not to exceed unity in absolute value.

***747.** Prove that if $a_0 \geqslant a_1 \geqslant a_2 \geqslant \ldots \geqslant a_n \geqslant 0$, then all the roots of the polynomial $f(x)=a_0x^n+a_1x^{n-1}+\ldots+a_n$ do not exceed unity in absolute value.

Sec. 10. Approximating Roots of a Polynomial

748. Compute to within 0.0001 the root of the equation $x^3-3x^2-13x-7=0$ which lies in the interval $(-1, 0)$.

749. Compute the real root of the equation $x^3-2x-5=0$ with an accuracy of 0.000001.

750. Compute the real roots of the following equations to within 0.0001:

(a) $x^3-10x-5=0$, (b) $x^3+2x-30=0$,

(c) $x^3-3x^2-4x+1=0$, (d) $x^3-3x^2-x+2=0$.

751. Divide a hemisphere of radius 1 into two equal parts by a plane parallel to the base.

752. Evaluate the positive root of the equation $x^3-5x-3=0$ to within 0.0001.

753. Compute to within 0.0001 the root of the equation:

(a) $x^4+3x^3-9x-9=0$ lying in the interval $(1, 2)$;

(b) $x^4-4x^3+4x^2-4=0$ lying in the interval $(-1, 0)$;

(c) $x^4+3x^3+4x^2+x-3=0$ lying in the interval $(0, 1)$;

(d) $x^4-10x^2-16x+5=0$ lying in the interval $(0, 1)$;

(e) $x^4-x^3-9x^2+10x-10=0$ lying in the interval $(-4, -3)$;

(f) $x^4-6x^2+12x-8=0$ lying in the interval $(1, 2)$;

(g) $x^4-3x^2+4x-3=0$ lying in the interval $(-3, -2)$;

(h) $x^4-x^3-7x^2-8x-6=0$ lying in the interval $(3, 4)$;

(i) $x^4-3x^3+3x^2-2=0$ lying in the interval $(1, 2)$.

754. Compute to within 0.0001 the real roots of the following equations:

(a) $x^4+3x^3-4x-1=0$,

(b) $x^4+3x^3-x^2-3x+1=0$,

(c) $x^4-6x^3+13x^2-10x+1=0$,

(d) $x^4-8x^3-2x^2+16x-3=0$,

(e) $x^4-5x^3+9x^2-5x-1=0$,

(f) $x^4-2x^3-6x^2+4x+4=0$,

(g) $x^4+2x^3+3x^2+2x-2=0$,

(h) $x^4+4x^3-4x^2-16x-8=0$.

CHAPTER 6
SYMMETRIC FUNCTIONS

Sec. 1. Expressing Symmetric Functions in Terms of Elementary Symmetric Functions. Computing Symmetric Functions of the Roots of an Algebraic Equation

755. Express the following in terms of the elementary symmetric polynomials:

(a) $x_1^3+x_2^3+x_3^3-3x_1x_2x_3$,

(b) $x_1^2x_2+x_1x_2^2+x_1^2x_3+x_1x_3^2+x_2^2x_3+x_2x_3^2$,

(c) $x_1^4+x_2^4+x_3^4-2x_1^2x_2^2-2x_2^2x_3^2-2x_3^2x_1^2$,

(d) $x_1^5x_2^2+x_1^2x_2^5+x_1^5x_3^2+x_1^2x_3^5+x_2^5x_3^2+x_2^2x_3^5$,

(e) $(x_1+x_2)(x_1+x_3)(x_2+x_3)$,

(f) $(x_1^2+x_2^2)(x_1^2+x_3^2)(x_2^2+x_3^2)$,

(g) $(2x_1-x_2-x_3)(2x_2-x_1-x_3)(2x_3-x_1-x_2)$,

(h) $(x_1-x_2)^2(x_1-x_3)^2(x_2-x_3)^2$.

756. Represent the following in terms of the elementary symmetric polynomials:

(a) $(x_1+x_2)(x_1+x_3)(x_1+x_4)(x_2+x_3)(x_2+x_4)(x_3+x_4)$,

(b) $(x_1x_2+x_3x_4)(x_1x_3+x_2x_4)(x_1x_4+x_2x_3)$,

(c) $(x_1+x_2-x_3-x_4)(x_1-x_2+x_3-x_4)(x_1-x_2-x_3+x_4)$.

757. Represent the following monogenic polynomials in terms of the elementary symmetric polynomials:

(a) $x_1^2+\cdots$,
(b) $x_1^3+\cdots$,
(c) $x_1^2x_2x_3+\cdots$,
(d) $x_1^2x_2^2+\cdots$,
(e) $x_1^3x_2+\cdots$,
(f) $x_1^4+\cdots$,
(g) $x_1^2x_2^2x_3+\cdots$,
(h) $x_1^3x_2x_3+\cdots$,
(i) $x_1^3x_2^2+\cdots$,
(j) $x_1^4x_2+\cdots$,
(k) $x_1^5+\cdots$,
(l) $x_1^2x_2^2x_3x_4+\ldots$,
(m) $x_1^2x_2^2x_3^2+\cdots$,
(n) $x_1^3x_2x_3x_4+\ldots$,
(o) $x_1^3x_2^2x_3+\cdots$,
(p) $x_1^3x_2^3+\cdots$,
(q) $x_1^4x_2x_3+\cdots$,
(r) $x_1^4x_2^2+\cdots$,
(s) $x_1^5x_2+\cdots$,
(t) $x_1^9+\cdots$

758. Express the following in terms of the elementary symmetric polynomials:

(a) $(-x_1+x_2+x_3+\ldots+x_n)^2+(x_1-x_2+x^3+\ldots+x_n)^2+(x_1+x_2-x_3+\ldots+x_n)^2+\ldots+(x_1+x_2+x_3+\ldots-x_n)^2$,

(b) $(-x_1+x_2+x_3+\ldots+x_n)\,(x_1-x_2+x_3+\ldots+x_n)\ldots(x_1+x_2+\ldots-x_n)$.

759. Express the following in terms of the elementary symmetric polynomials:

(a) $\sum\limits_{i>k}(x_i-x_k)^2$, (b) $\sum\limits_{i>k}(x_i+x_k)^3$,

(c) $\sum\limits_{i>k}(x_i-x_k)^4$, (d) $\sum\limits_{\substack{i>k\\ j\neq i;\ j\neq k}}(x_i+x_k-x_j)^2$.

760. Express the following monogenic polynomial in terms of the elementary symmetric polynomials:

$$x_1^2x_2^2\ \cdots\ x_k^2+\ \cdots$$

761. Express the following in terms of the elementary symmetric polynomials:

$$\sum(a_1x_{i_1}+a_2x_{i_2}+\ \cdots\ +a_nx_{i_n})^2.$$

The sum is extended over all possible permutations $i_1, i_2, \ldots, i_n$ of the numbers 1, 2, ..., n.

762. Express the following in terms of the elementary symmetric polynomials:

(a) $\dfrac{x_1}{x_2}+\dfrac{x_2}{x_3}+\dfrac{x_3}{x_1}+\dfrac{x_2}{x_1}+\dfrac{x_3}{x_2}+\dfrac{x_1}{x_3}$,

(b) $\dfrac{(x_1-x_2)^2}{x_1+x_2}+\dfrac{(x_2-x_3)^2}{x_2+x_3}+\dfrac{(x_3-x_1)^2}{x_3+x_1}$,

(c) $\left(\dfrac{x_2}{x_1}+\dfrac{x_3}{x_2}+\dfrac{x_1}{x_3}\right)\left(\dfrac{x_1}{x_2}+\dfrac{x_2}{x_3}+\dfrac{x_3}{x_1}\right)$.

763. Express the following in terms of the elementary symmetric polynomials:

(a) $\dfrac{x_1x_2}{x_3x_4}+\dfrac{x_1x_3}{x_2x_4}+\dfrac{x_1x_4}{x_2x_3}+\dfrac{x_2x_3}{x_1x_4}+\dfrac{x_2x_4}{x_1x_3}+\dfrac{x_3x_4}{x_1x_2}$,

(b) $\dfrac{x_1+x_2}{x_3+x_4}+\dfrac{x_1+x_3}{x_2+x_4}+\dfrac{x_1+x_4}{x_2+x_3}+\dfrac{x_2+x_3}{x_1+x_4}+\dfrac{x_2+x_4}{x_1+x_3}+\dfrac{x_3+x_4}{x_1+x_2}$.

764. Express the following in terms of the elementary symmetric polynomials:

(a) $\sum \frac{1}{x_i}$, (b) $\sum \frac{1}{x_i^2}$, (c) $\sum\limits_{i \neq j} \frac{x_i}{x_j}$,

(d) $\sum\limits_{i \neq j} \frac{x_i^2}{x_j}$, (e) $\sum\limits_{i \neq j} \frac{x_i^2}{x_j}$, (f) $\sum\limits_{\substack{i \neq j \\ i \neq k \\ j > k}} \frac{x_j x_k}{x_i}$.

765. Compute the sum of the squares of the roots of the equation

$$x^3+2x-3=0.$$

766. Compute $x_1^3x_2+x_1x_2^3+x_2^3x_3+x_2x_3^3+x_3^3x_1+x_3x_1^3$ of the roots of the equation $x^3-x^2-4x+1=0$.

767. Determine the value of the monogenic symmetric function

$$x_1^3x_2x_3+\ldots$$

of the roots of the equation

$$x^4+x^3-2x^2-3x+1=0.$$

768. Let x_1, x_2, x_3 be the roots of the equation $x^3+px+q=0$. Compute:

(a) $\frac{x_1}{x_2}+\frac{x_2}{x_3}+\frac{x_3}{x_1}+\frac{x_2}{x_1}+\frac{x_3}{x_2}+\frac{x_1}{x_3}$,

(b) $x_1^4x_2^2+x_1^2x_2^4+x_2^4x_3^2+x_2^2x_3^4+x_3^4x_1^2+x_3^2x_1^4$,

(c) $(x_1^2-x_2x_3)(x_2^2-x_1x_3)(x_3^2-x_2x_1)$,

(d) $(x_1+x_2)^4(x_1+x_3)^4(x_2+x_3)^4$,

(e) $\frac{x_1^2}{(x_2+1)(x_3+1)}+\frac{x_2^2}{(x_1+1)(x_3+1)}+\frac{x_3^2}{(x_1+1)(x_2+1)}$,

(f) $\frac{x_1^2}{(x_1+1)^2}+\frac{x_2^2}{(x_2+1)^2}+\frac{x_3^2}{(x_3+1)^2}$.

769. What relationship is there between the coefficients of the cubic equation

$$x^3+ax^2+bx+c=0$$

if the square of one of the roots is equal to the sum of the squares of the other two?

770. Prove the following theorem: for all roots of the cubic equation $x^3+ax^2+bx+c=0$ to have negative real parts, it is necessary and sufficient that the following conditions hold:

$$a>0,\ ab-c>0,\ c>0.$$

771. Find the area and the radius of a circle circumscribed about a triangle whose sides are equal to the roots of the cubic equation

$$x^3-ax^2+bx-c=0.$$

***772.** Find the relationship among the coefficients of an equation whose roots are equal to the sines of the angles of a triangle.

773. Compute the value of a symmetric function of the roots of the equation $f(x)=0$:

(a) $x_1^4x_2+ \ \ldots, \quad f(x)=3x^3-5x^2+1;$

(b) $x_1^3x_2^3+ \ \ldots, \quad f(x)=3x^4-2x^3+2x^2+x-1;$

(c) $(x_1^2+x_1x_2+x_2^2)(x_2^2+x_2x_3+x_3^2)(x_3^2+x_3x_1+x_1^2),$

$$f(x)=5x^3-6x^2+7x-8.$$

774. Express in terms of the coefficients of the equation

$$a_0x^3+a_1x^2+a_2x+a_3=0$$

the following symmetric functions:

(a) $a_0^4(x_1-x_2)^2(x_1-x_3)^2(x_2-x_3)^2,$

(b) $a_0^4(x_1^2-x_2x_3)(x_2^2-x_1x_3)(x_3^2-x_1x_2),$

(c) $\dfrac{(x_1-x_2)^2}{x_1x_2}+\dfrac{(x_1-x_3)^2}{x_1x_3}+\dfrac{(x_2-x_3)^2}{x_2x_3},$

(d) $a_0^4(x_1^2+x_1x_2+x_2^2)(x_2^2+x_2x_3+x_3^2)(x_3^2+x_3x_1+x_1^2).$

775. Let $x_1, x_2, \ldots, x_n$ be the roots of the polynomial

$$x^n+a_1x^{n-1}+ \ \cdots \ +a_n.$$

Prove that the symmetric polynomial in $x_2, x_3, \ldots, x_n$ can be represented in the form of a polynomial in x_1.

776. Using the result of Problem 775, solve Problems 755(e), 755 (g), 774 (b), 774 (d).

777. Find $\sum\limits_{i=1}^{n}\dfrac{\partial f_k}{\partial x_i}$ where f_k is the kth elementary symmetric function of $x_1, x_2, \ldots, x_n$.

778. Let the expression of the symmetric function $F(x_1, x_2, \ldots, x_n)$ in terms of the elementary symmetric functions be known. Find the expression of $\sum\limits_{i=1}^{n}\dfrac{\partial F}{\partial x_i}$ in terms of the elementary symmetric functions.

Prove the theorems:

779. If $F(x_1, x_2, \ldots, x_n)$ is a symmetric function having the property

$$F(x_1+a, x_2+a, \ldots, x_n+a)=F(x_1, x_2, \ldots, x_n)$$

and if $\Phi(f_1, f_2, \ldots, f_n)$ is its expression in terms of the elementary symmetric functions, then

$$n\frac{\partial\Phi}{\partial f_1}+(n-1)f_1\frac{\partial\Phi}{\partial f_2}+\cdots+f_{n-1}\frac{\partial\Phi}{\partial f_n}=0$$

and conversely.

780. Every homogeneous symmetric polynomial of degree two having the property of Problem 779 is equal to $\alpha\sum\limits_{i<k}(x_i-x_k)^2$ where α is a constant.

781. Find the general form of homogeneous symmetric polynomials of degree three having the property of Problem 779.

782. Using the result of Problem 779, express the following in terms of the elementary symmetric polynomials:

$$\sum_{i<j<k}(x_i-x_j)^2(x_i-x_k)^2(x_j-x_k)^2.$$

783. Prove that among the symmetric polynomials $F(x_1, x_2, \ldots, x_n)$ having the property

$$F(x_1, x_2, \ldots, x_n)=F(x_1+a, x_2+a, \ldots, x_n+a)$$

there are $n-1$ "elementary polynomials" $\varphi_2, \varphi_3, \ldots, \varphi_n$ such that each polynomial of the class under consideration can be expressed in the form of a polynomial in $\varphi_2, \varphi_3, \ldots, \varphi_n$.

784. Express the following symmetric functions in terms of the polynomials φ_2, φ_3 of Problems 783:

(a) $(x_1-x_2)^2(x_1-x_3)^2(x_2-x_3)^2$,

(b) $(x_1-x_2)^4+(x_1-x_3)^4+(x_2-x_3)^4$.

785. Express the following symmetric functions in terms of the polynomials $\varphi_2, \varphi_3, \varphi_4$ of Problem 783:

(a) $(x_1+x_2-x_3-x_4)(x_1-x_2+x_3-x_4)(x_1-x_2-x_3+x_4)$,

(b) $(x_1-x_2)^2(x_1-x_3)^2(x_1-x_4)^2(x_2-x_3)^2(x_2-x_4)^2(x_3-x_4)^2$.

Sec. 2. Power Sums

786. Find an expression for s_2, s_3, s_4, s_5, s_6 in terms of the elementary symmetric polynomials, using Newton's formulas.

787. Express f_2, f_3, f_4, f_5, f_6 in terms of the power sums $s_1, s_2, \ldots$, using Newton's formulas.

788. Find the sum of the fifth powers of the roots of the equation

$$x^6-4x^5+3x^3-4x^2+x+1=0.$$

789. Find the sum of the eighth powers of the roots of the equation

$$x^4-x^3-1=0.$$

790. Find the sum of the tenth powers of the roots of the equation

$$x^3-3x+1=0.$$

791. Find $s_1, s_2, \ldots, s_n$ of the roots of the equation

$$x^n+\frac{x^{n-1}}{1}+\frac{x^{n-2}}{1\cdot 2}+\ \cdots\ +\frac{1}{n!}=0.$$

792. Prove that

$$a^k(x_1^k+x_2^k)=(-1)^k\left[b^k-\frac{k}{1}b^{k-2}ac+\frac{k(k-3)}{1\cdot 2}b^{k-4}a^2c^2\right.$$
$$\left.-\frac{k(k-4)(k-5)}{1\cdot 2\cdot 3}b^{k-6}a^3c^3+\ \cdots\right]$$

if x_1, x_2 are the roots of the quadratic equation $ax^2+bx+c=0$.

793. Prove that for any cubic equation

$$\frac{s_1^5-s_5}{s_1^3-s_3}=\frac{5}{3}(f_1^2-f_2).$$

794. Prove that if the sum of the roots of a quartic equation is equal to zero, then

$$\frac{s_5}{5}=\frac{s_3}{3}\cdot\frac{s_2}{2}.$$

795. Prove that if $s_1=s_3=0$ for a sextic equation, then

$$\frac{s_7}{7}=\frac{s_5}{5}\cdot\frac{s_2}{2}.$$

796. Find nth-degree equations for which

$$s_1=s_2=\ldots=s_{n-1}=0.$$

797. Find nth-degree equations for which

$$s_2=s_3=\ldots=s_n=0.$$

798. Find an nth-degree equation for which

$$s_2=1,\ s_3=s_4=\ldots=s_n=s_{n+1}=0.$$

799. Express $\sum\limits_{i<j} x_i^k x_j^k$ in terms of power sums.

***800.** Express $\sum\limits_{i<j} (x_i+x_j)^k$ in terms of power sums.

***801.** Express $\sum\limits_{i<j} (x_i-x_j)^{2k}$ in terms of power sums.

802. Prove that $s_k=\begin{vmatrix} f_1 & 1 & 0 & \cdots & 0 \\ 2f_2 & f_1 & 1 & \cdots & 0 \\ 3f_3 & f_2 & f_1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ kf_k & f_{k-1} & f_{k-2} & \cdots & f_1 \end{vmatrix}$.

803. Prove that $f_k=\dfrac{1}{k!}\begin{vmatrix} s_1 & 1 & 0 & \cdots & 0 \\ s_2 & s_1 & 2 & \cdots & 0 \\ s_3 & s_2 & s_1\ 3 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ s_k & s_{k-1} & s_{k-2} & \cdots & s_1 \end{vmatrix}$.

804. Compute the determinant $\begin{vmatrix} x^n & x^{n-1} & x^{n-2} & \cdots & 1 \\ s_1 & 1 & 0 & \cdots & 0 \\ s_2 & s_1 & 2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ s_n & s_{n-1} & s_{n-2} & \cdots & n \end{vmatrix}$.

***805.** Find s_m of the roots of the equation

$$X_n(x)=0.$$

***806.** Prove that f_2, f_3 and f_4 of the roots of the equation $X_n(x)=$ $=0$ can only take on values 0 and ± 1.

***807.** Solve the system of equations

$$\begin{aligned} x_1+x_2+ \cdots +x_n&=a, \\ x_1^2+x_2^2+ \cdots +x_n^2&=a, \\ \cdots\cdots&\cdots\cdots \\ x_1^n+x_2^n+ \cdots +x_n^n&=a \end{aligned}$$

and find $x_1^{n+1}+x_2^{n+1}+ \cdots +x_n^{n+1}$.

***808.** Compute the power sums $s_1, s_2, \ldots, s_n$ of the roots of the equation

$$x^n+(a+b)x^{n-1}+(a^2+ab+b^2)x^{n-2}+\cdots+(a^n+a^{n-1}b+\cdots+b^n)=0.$$

***809.** Compute the power sums $s_1, s_2, \ldots, s_n$ of the roots of the equation

$$x^n+(a+b)x^{n-1}+(a^2+b^2)x^{n-2}+\cdots+(a^n+b^n)=0.$$

Sec. 3. Transformation of Equations

810. Find equations whose roots are:

(a) $x_1+x_2,\ x_2+x_3,\ x_3+x_1$;

(b) $(x_1-x_2)^2,\ (x_2-x_3)^2,\ (x_3-x_1)^2$;

(c) $x_1^2-x_2x_3,\ x_2^2-x_3x_1,\ x_3^2-x_1x_2$;

(d) $(x_1-x_2)(x_1-x_3),\ (x_2-x_1)(x_2-x_3),\ (x_3-x_1)(x_3-x_2)$;

(e) $x_1^2,\ x_2^2,\ x_3^2$; (f) $x_1^3,\ x_2^3,\ x_3^3$

where x_1, x_2, x_3 are the roots of the equation $x^3+ax^2+bx+c=0$.

811. Find an equation whose roots are

$$(x_1+x_2\varepsilon+x_3\varepsilon^2)^3 \text{ and } (x_1+x_2\varepsilon^2+x_3\varepsilon)^3$$

where $\varepsilon=-\frac{1}{2}+i\frac{\sqrt{3}}{2}$; x_1, x_2, x_3 are the roots of the equation

$$x^3+ax^2+bx+c=0.$$

812. Find an equation of lowest degree, one of the roots of which is $\frac{x_1}{x_2}+\frac{x_2}{x_3}+\frac{x_3}{x_1}$, where x_1, x_2, x_3 are the roots of the cubic equation $x^3+ax^2+bx+c=0$, and the coefficients of which are expressed rationally in terms of the coefficients of the given equation.

813. Find an equation of lowest degree, one of the roots of which is $\frac{x_1}{x_2}$ where x_1, x_2, x_3 are the roots of the equation $x^3+ax^2+bx+c=0$, and the coefficients of which are expressed in terms of the coefficients of the given equation.

814. Find an equation of lowest degree with coefficients expressed rationally in terms of the coefficients of a given equation

$x^4+ax^3+bx^2+cx+d=0$, one of the roots of the desired equation being:

(a) $x_1x_2+x_3x_4$, (b) $(x_1+x_2-x_3-x_4)^2$, (c) x_1x_2,

(d) x_1+x_2, (e) $(x_1-x_2)^2$.

815. Using the results of Problems 814 (a) and 814 (b), express the roots of a quartic equation in terms of roots of the auxiliary cubic equation of Problem 814 (a).

816. Write a formula for the solution of equation

$$x^4-6ax^2+bx-3a^2=0.$$

817. Write an equation, one of the roots of which is

$$(x_1x_2+x_2x_3+x_3x_4+x_4x_5+x_5x_1)$$
$$\times(x_1x_3+x_3x_5+x_5x_2+x_2x_4+x_4x_1)$$

where x_1, x_2, x_3, x_4, x_5 are roots of the equation

$$x^5+ax+b=0.$$

Sec. 4. Resultant and Discriminant

***818.** Prove that the resultant of the polynomials

$$f(x)=x^n+a_1x^{n-1}+\ldots+a_n \text{ and } \varphi(x)=b_0x^m+\ldots+b_m$$

is equal to a determinant made up of the coefficients of the remainders left after dividing $\varphi(x)$, $x\varphi(x)$, ..., $x^{n-1}\varphi(x)$ by $f(x)$. It is assumed that the remainders are arranged in order of increasing powers of x (Hermite's method).

Remark. The remainder $r_k(x)$ left after dividing $x^{k-1}\varphi(x)$ by $f(x)$ is equal to the remainder obtained upon division of $xr_{k-1}(x)$ by $f(x)$.

***819.** Prove that the resultant of the polynomials

$$f(x)=a_0x^n+a_1x^{n-1}+\ldots+a_n$$

and

$$\varphi(x)=b_0x^n+b_1x^{n-1}+\ldots+b_n$$

is equal to a determinant composed of the coefficients of polynomials of degree $n-1$ (or lower)

$$\psi_k(x)=(a_0x^{k-1}+a_1x^{k-2}+\cdots+a_{k-1})\varphi(x)$$
$$-(b_0x^{k-1}+b_1x^{k-2}+\cdots+b_{k-1})f(x)$$

$k=1, \ldots, n$ (Bézout's method).

Remark. $\psi_1 = a_0\varphi - b_0 f$,

$$\psi_k = x\psi_{k-1} + a_{k-1}\varphi - b_{k-1}f.$$

***820.** Prove that the resultant of the polynomials

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$$

and

$$\varphi(x) = b_0x^m + b_1x^{m-1} + \cdots + b_m$$

is equal, for $n > m$, to a determinant made up of the coefficients of polynomials $\chi_k(x)$ of degree not exceeding $n-1$ determined from the formulas

$$\chi_k(x) = x^{k-1}\varphi(x) \text{ for } 1 \leqslant k \leqslant n-m,$$

$$\chi_k(x) = (a_0x^{k-n+m-1} + a_1x^{k-n+m-2} + \cdots + a_{k-n+m-1})\, x^{n-m}\varphi(x) - (b_0x^{k-n+m-1} + b_1x^{k-n+m-2} + \cdots + b_{k-n+m-1})f(x)$$

(the polynomials χ_k are arranged in order of increasing powers of x).

Remark. $\chi_{n-m+1} = a_0x^{n-m}\varphi(x) - b_0f(x)$,

$$\chi_k = x\chi_{k-1} + a_{k-n+m-1}\, x^{n-m}\varphi(x) - b_{k-n+m-1}f(x)$$

for $k > n-m+1$.

821. Compute the resultant of the polynomials:

(a) $x^3 - 3x^2 + 2x + 1$ and $2x^2 - x - 1$;

(b) $2x^3 - 3x^2 + 2x + 1$ and $x^2 + x + 3$;

(c) $2x^3 - 3x^2 - x + 2$ and $x^4 - 2x^2 - 3x + 4$;

(d) $3x^3 + 2x^2 + x + 1$ and $2x^3 + x^2 - x - 1$;

(e) $2x^4 - x^3 + 3$ and $3x^3 - x^2 + 4$;

(f) $a_0x^2 + a_1x + a_2$ and $b_0x^2 + b_1x + b_2$.

822. For what value of λ do the following polynomials have a common root:

(a) $x^3 - \lambda x + 2$ and $x^2 + \lambda x + 2$;

(b) $x^3 - 2\lambda x + \lambda^3$ and $x^2 + \lambda^2 - 2$;

(c) $x^3 + \lambda x^2 - 9$ and $x^3 + \lambda x - 3$?

823. Eliminate x from the following systems of equations:

(a) $x^2 - xy + y^2 = 3$, $x^2y + xy^2 = 6$;

(b) $x^3-xy-y^3+y=0,\qquad x^2+x-y^2-1=0;$

(c) $y=x^3-2x^2-6x+8,\qquad y=2x^3-8x^2+5x+2.$

824. Solve the following systems:

(a) $y^2-7xy+4x^2+13x-2y-3=0,$
$y^2-14xy+9x^2+28x-4y-5=0;$

(b) $y^2+x^2-y-3x=0,$
$y^2-6xy-x^2+11y+7x-12=0;$

(c) $5y^2-6xy+5x^2-16=0,$
$y^2-xy+2x^2-y-x-4=0;$

(d) $y^2+(x-4)y+x^2-2x+3=0,$
$y^3-5y^2+(x+7)y+x^3-x^2-5x-3=0;$

(e) $2y^3-4xy^2-(2x^2-12x+8)y+x^3+6x^2-16x=0,$
$4y^3-(3x+10)y^2-(4x^2-24x+16)y-3x^3+2x^2-12x+40=0.$

825. Determine the resultant of the polynomials

$$a_0x^n+a_1x^{n-1}+\cdots+a_n$$

and

$$a_0x^{n-1}+a_1x^{n-2}+\cdots+a_{n-1}.$$

826. Prove that $\Re(f,\ \varphi_1\cdot\varphi_2)=\Re(f,\ \varphi_1)\cdot\Re(f,\ \varphi_2)$.

***827.** Find the resultant of the polynomials

$$X_n \text{ and } x^m-1.$$

***828.** Find the resultant of the polynomials X_m and X_n.

829. Compute the discriminant of the polynomial:

(a) $x^3-x^2-2x+1,\qquad$ (b) $x^3+2x^2+4x+1,$

(c) $3x^3+3x^2+5x+2,\qquad$ (d) $x^4-x^3-3x^2+x+1,$

(e) $2x^4-x^3-4x^2+x+1.$

830. Compute the discriminant of the polynomial:

(a) $x^5-5ax^3+5a^2x-b,\qquad$ (b) $(x^2-x+1)^3-\lambda(x^2-x)^2,$

(c) $ax^3-bx^2+(b-3a)x+a,$

(d) $x^4-\lambda x^3+3(\lambda-4)x^2-2(\lambda-8)x-4.$

831. For what value of λ does the polynomial have multiple roots:

(a) $x^3-3x+\lambda$; (b) $x^4-4x+\lambda$,

(c) $x^3-8x^2+(13-\lambda)x-(6+2\lambda)$,

(d) $x^4-4x^3+(2-\lambda)x^2+2x-2$?

832. Characterize the number of real roots of a polynomial with real coefficients by the sign of the discriminant:
(a) for a third-degree polynomial;
(b) for a fourth-degree polynomial;
(c) in the general case.

833. Compute the discriminant of the polynomial x^n+a.

***834.** Compute the discriminant of the polynomial x^n+px+q.

***835.** Compute the discriminant of the polynomial

$$a_0x^{m+n}+a_1x^m+a_2.$$

836. Knowing the discriminant of the polynomial

$$a_0x^n+a_1x^{n-1}+\cdots+a_n$$

find the discriminant of the polynomial

$$a_nx^n+a_{n-1}x^{n-1}+\cdots+a_0.$$

837. Prove that the discriminant of a fourth-degree polynomial is equal to the discriminant of its Ferrari resolvent (Problem 814(a) and Problem 80).

838. Prove that

$$D((x-a)f(x))=D(f(x))\,[f(a)]^2.$$

***839.** Compute the discriminant of the polynomial

$$x^{n-1}+x^{n-2}+\cdots+1.$$

***840.** Compute the discriminant of the polynomial

$$x^n+ax^{n-1}+ax^{n-2}+\cdots+a.$$

841. Prove that the discriminant of a product of two polynomials is equal to the product of the discriminants multiplied by the square of their resultant.

842. Find the discriminant of the polynomial

$$X_{p^m}=\frac{x^{p^m}-1}{x^{p^{m-1}}-1}.$$

***843.** Find the discriminant of the cyclotomic polynomial X_n.

***844.** Compute the discriminant of the polynomial

$$E_n = n!\left(1+\frac{x}{1}+\frac{x^2}{1\cdot 2}+\cdots+\frac{x^n}{n!}\right).$$

***845.** Compute the discriminant of the polynomial

$$F_n = x^n+\frac{a}{1}x^{n-1}+\frac{a(a-1)}{1\cdot 2}x^{n-2}+\cdots+\frac{a(a-1)\cdots(a-n+1)}{n!}.$$

***846.** Compute the discriminant of the Hermite polynomial

$$P_n(x) = (-1)^n e^{\frac{x^2}{2}}\frac{d^n e^{-\frac{x^2}{2}}}{dx^n}.$$

***847.** Compute the discriminant of the Laguerre polynomial

$$P_n(x) = (-1)^n\frac{d^n(x^n e^{-x})}{dx^n}.$$

***848.** Compute the discriminant of the Chebyshev polynomial

$$2\cos\left(n\arccos\frac{x}{2}\right).$$

***849.** Compute the discriminant of the polynomial

$$P_n(x) = \frac{(-1)^n}{n!}(1+x^2)^{n+1}\frac{d^n\left(\frac{1}{1+x^2}\right)}{dx^n}.$$

***850.** Compute the discriminant of the polynomial

$$P_n(x) = (-1)^n(1+x^2)^{n+\frac{1}{2}}\frac{d^n\frac{1}{\sqrt{1+x^2}}}{dx^n}.$$

***851.** Compute the discriminant of the polynomial

$$P_n(x) = (-1)^n x^{2n+2}e^{-\frac{1}{x}}\frac{d^{n+1}\left(e^{\frac{1}{x}}\right)}{dx^{n+1}}.$$

***852.** Find the maximum of the discriminant of the polynomial

$$x^n+a_1x^{n-1}+\cdots+a_n$$

all the roots of which are real and connected by the relation

$$x_1^2+x_2^2+\ldots+x_n^2 = n(n-1)R^2.$$

853. Knowing the discriminant of $f(x)$, find the discriminant of $f(x^2)$.

854. Knowing the discriminant of $f(x)$, find the discriminant of $f(x^m)$.

855. Prove that the discriminant of $F(x)=f(\varphi(x))$ is equal to

$$[D(f)]^m \prod_{i=1}^{n} D\left(\varphi(x)-x_i\right),$$

where m is the degree of $\varphi(x)$; $x_1, x_2, \ldots, x_n$ are roots of $f(x)$. The leading coefficients of f and φ are taken equal to unity.

Sec. 5. The Tschirnhausen Transformation and Rationalization of the Denominator

856. Transform the equation $(x-1)(x-3)(x+4)=0$ by the substitution $y=x^2-x-1$.

857. Transform the following equations:

(a) $x^3-3x-4=0$ by the substitution $y=x^2+x+1$;

(b) $x^3+2x^2+2=0$ by the substitution $y=x^2+1$;

(c) $x^4-x-2=0$ by the substitution $y=x^3-2$;

(d) $x^4-x^3-x^2+1=0$ by the substitution $y=x^3+x^2+x+1$.

858. Transform the following equations by the Tschirnhausen transformation and find the inverse transformations:

(a) $x^3-x+2=0$, $y=x^2+x$;

(b) $x^4-3x+1=0$, $y=x^3+x$;

(c) $x^4+5x^3+6x^2-1=0$, $y=x^3+4x^2+3x-1$.

859. Transform the equation $x^3-x^2-2x+1=0$ by the substitution $y=2-x^2$ and interpret the result.

860. Prove that for the roots of a cubic equation with rational coefficients to be expressed rationally with rational coefficients in terms of one another, it is necessary and sufficient that the discriminant be the square of a rational number.

861. Rationalize the denominators:

(a) $\dfrac{1}{1+\sqrt{2}-\sqrt{3}}$, (b) $\dfrac{1}{1+\sqrt[3]{2}+2\sqrt[3]{4}}$, (c) $\dfrac{7}{1-\sqrt[4]{2}+\sqrt{2}}$.

862. Rationalize the denominators:

(a) $\dfrac{\alpha}{\alpha+1}$, $\alpha^3-3\alpha+1=0$;

(b) $\dfrac{\alpha^2-3\alpha-1}{\alpha^2+2\alpha+1}$, $\alpha^3+\alpha^2+3\alpha+4=0$;

(c) $\frac{1}{3\alpha^3+\alpha^2-2\alpha-1}$, $\alpha^4-\alpha^3+2\alpha+1=0$;

(d) $\frac{1}{\alpha^3+3\alpha^2+3\alpha+2}$, $\alpha^4+\alpha^3-4\alpha^2-3\alpha+2=0$.

863. Prove that every rational function of a root x_1 of the cubic equation $x^3+ax^2+bx+c=0$ can be represented as $\frac{Ax_1+B}{Cx_1+D}$ with coefficients A, B, C, D, which can be expressed rationally in terms of the coefficients of the original expression and in terms of the coefficients a, b, c.

864. Let the discriminant of a cubic equation that has rational coefficients and is irreducible over the field of rationals be the square of a rational number. It is then possible to establish the relation $x_2=\frac{\alpha x_1+\beta}{\gamma x_1+\delta}$ among the roots. What condition do the coefficients α, β, γ, δ have to satisfy?

865. Make the transformation $y=x^2$ in the equation

$$a_0x^n+a_1x^{n-1}+\cdots+a_n=0.$$

866. Make the transformation $y=x^3$ in the equation

$$a_0x^n+a_1x^{n-1}+\cdots+a_n=0.$$

***867.** Prove that if all the roots x_i of the polynomial

$$f(x)=x^n+a_1x^{n-1}+\cdots+a_n, \quad a_n\neq 0$$

with integral coefficients satisfy the condition $|x_i|\leqslant 1$, then they are all roots of unity.

Sec. 6. Polynomials that Remain Unchanged under Even Permutations of the Variables. Polynomials that Remain Unchanged under Circular Permutations of the Variables

868. Prove that if a polynomial remains unchanged under even permutations and changes sign under odd permutations, then it is divisible by the Vandermonde determinant made up of the variables, and the quotient is a symmetric polynomial.

869. Prove that every polynomial that remains unchanged under even permutations of the variables can be represented as

$$F_1+F_2\,\Delta$$

where F_1 and F_2 are symmetric polynomials and Δ is the Vandermonde determinant made up of the variables.

870. Evaluate

$$\begin{vmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-2} & x_1^{n+1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-2} & x_2^{n+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-2} & x_n^{n+1} \end{vmatrix}.$$

871. Form an equation whose roots are $\alpha x_1+\beta x_2+\gamma x_3$, $\alpha x_2+\beta x_3+\gamma x_1$ and $\alpha x_3+\beta x_1+\gamma x_2$ where x_1, x_2, x_3 are the roots of the equation $x^3+ax^2+bx+c=0$.

872. Form an equation whose roots are $x_1y_1+x_2y_2+x_3y_3$, $x_1y_2+x_2y_3+x_3y_1$, $x_1y_3+x_2y_1+x_3y_2$ where x_1, x_2, x_3 are the roots of the equation $x^3+px+q+0$, and y_1, y_2, y_3 are the roots of the equation $y^3+p'y+q'=0$.

873. For the following equations with rational coefficients

$$x^3+px+q=0,$$
$$y^3+p'y+q'=0$$

to be connected by a rational Tschirnhausen transformation, it is necessary and sufficient that the ratio of their discriminants Δ and Δ' be the square of a rational number and that one of the equations

$$u^3=3pp'u+\frac{27qq'\pm\sqrt{\Delta\Delta'}}{2}$$

have a rational root. Prove this.

874. Prove that every polynomial in n variables $x_1, x_2, \ldots, x_n$ which remains unchanged under circular permutations of the variables may be represented as

$$\sum Af_1^{\alpha_0}\eta_1^{\alpha_1}\eta_2^{\alpha_2}\ \cdots\ \eta_{n-1}^{\alpha_{n-1}},$$

where $\eta_1, \eta_2, \ldots, \eta_{n-1}$ are linear forms:

$$\begin{aligned} \eta_1 &= x_1\varepsilon + x_2\varepsilon^2 + \cdots + x_n, \\ \eta_2 &= x_1\varepsilon^2 + x_2\varepsilon^4 + \cdots + x_n, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ \eta_{n-1} &= x_1\varepsilon^{n-1} + x_2\varepsilon^{2n-2} + \cdots + x_n; \end{aligned}$$

$$\varepsilon=\cos\frac{2\pi}{n}+i\sin\frac{2\pi}{n}.$$

The exponents $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ satisfy the condition: n divides $\alpha_1+2\alpha_2+\ldots+(n-1)\ \alpha_{n-1}$.

875. For rational functions that do not change under circular permutations of the variables, indicate n elementary ones (fractional and with nonrational coefficients) in terms of which all of them can be expressed rationally.

876. For rational functions of three variables unaltered under circular permutations, indicate three elementary functions with rational coefficients.

877. For rational functions of four variables that remain unchanged under circular permutations, indicate four elementary functions with rational coefficients.

878. For rational functions of five variables that remain fixed under circular permutations, indicate five elementary functions with rational coefficients.

CHAPTER 7
LINEAR ALGEBRA

In this chapter we adhere to the following terminology and notations. The term *space* is used to denote a vector space over the field of real numbers, unless otherwise stated. This term is used both for the space as a whole and for any part of a larger space (the term *subspace* will be used only when it is necessary to specify that a given space is part of a larger space). A *linear manifold* is a set of vectors of the form X_0+X, where X_0 is some fixed vector and X runs through the set of all vectors of some subspace.

The equation $X=(x_1, x_2, \ldots, x_n)$ means that X has coordinates $x_1, x_2, \ldots, x_n$ in some fixed basis of the space; when we deal with Euclidean space, the basis is assumed to be orthogonally normalized.

Vectors are sometimes called points, one-dimensional manifolds are called straight lines, and two-dimensional manifolds are called planes.

Sec. 1. Subspaces and Linear Manifolds. Transformation of Coordinates

879. Given a vector space spanned by the vectors $X_1, X_2, \ldots, X_m$. Determine the basis and dimension:

(a) $X_1=(2, 1, 3, 1)$, $X_2=(1, 2, 0, 1)$,

$X_3=(-1, 1, -3, 0)$;

(b) $X_1=(2, 0, 1, 3, -1)$, $X_2=(1, 1, 0, -1, 1)$,

$X_3=(0, -2, 1, 5, -3)$, $X_4=(1, -3, 2, 9, -5)$;

(c) $X_1=(2, 1, 3, -1)$, $X_2=(-1, 1, -3, 1)$,

$X_3=(4, 5, 3, -1)$, $X_4=(1, 5, -3, 1)$.

880. Determine the basis and dimension of the union and intersection of spaces spanned by the vectors $X_1, \ldots, X_k$ and $Y_1, \ldots, Y_m$:

(a) $X_1=(1, 2, 1, 0)$, $Y_1=(2, -1, 0, 1)$,
$X_2=(-1, 1, 1, 1)$, $Y_2=(1, -1, 3, 7)$;

(b) $X_1=(1, 2, -1, -2)$, $Y_1=(2, 5, -6, -5)$,
$X_2=(3, 1, 1, 1)$, $Y_2=(-1, 2, -7, -3)$,
$X_3=(-1, 0, 1, -1)$;

(c) $X_1=(1, 1, 0, 0)$, $Y_1=(0, 0, 1, 1)$,
$X_2=(1, 0, 1, 1)$, $Y_2=(0, 1, 1, 0)$.

881. Find the coordinates of the vector X in the basis E_1, E_2, E_3, E_4:

(a) $X=(1, 2, 1, 1)$, $E_1=(1, 1, 1, 1)$,
$E_2=(1, 1, -1, -1)$,
$E_3=(1, -1, 1, -1)$, $E_4=(1, -1, -1, 1)$;

(b) $X=(0, 0, 0, 1)$, $E_1=(1, 1, 0, 1)$, $E_2=(2, 1, 3, 1)$,
$E_3=(1, 1, 0, 0)$, $E_4=(0, 1, -1, -1)$.

882. Develop formulas for the transformation of coordinates from the basis E_1, E_2, E_3, E_4 to the basis E_1', E_2', E_3', E_4':

(a) $E_1=(1, 0, 0, 0)$, $E_2=(0, 1, 0, 0)$, $E_3=(0, 0, 1, 0)$,
$E_4=(0, 0, 0, 1)$, $E_1'=(1, 1, 0, 0)$, $E_2'=(1, 0, 1, 0)$,
$E_3'=(1, 0, 0, 1)$, $E_4'=(1, 1, 1, 1)$;

(b) $E_1=(1, 2, -1, 0)$, $E_2=(1, -1, 1, 1)$,
$E_3=(-1, 2, 1, 1)$, $E_4=(-1, -1, 0, 1)$,
$E_1'=(2, 1, 0, 1)$,
$E_2'=(0, 1, 2, 2)$, $E_3'=(-2, 1, 1, 2)$,
$E_4'=(1, 3, 1, 2)$.

883. The equation of a "surface" with respect to some basis $E_1, \ldots, E_4$ has the form $x_1^2+x_2^2-x_3^2-x_4^2=1$. Find the equation of this surface relative to the basis

$E_1'=(1, 1, 1, 1)$, $E_3'=(1, -1, 1, -1)$,
$E_2'=(1, 1, -1, -1)$, $E_4'=(1, -1, -1, 1)$

(the coordinates are given in the same basis $E_1, \ldots, E_4$).

*884. In the space of polynomials in $\cos x$ of degree not exceeding n, write the formulas for transformation of coordinates from the basis $1, \cos x, \ldots, \cos^n x$ to the basis $1, \cos x, \ldots, \cos nx$, and conversely.

885. Find a straight line in four-dimensional space that passes through the origin of coordinates and intersects the straight lines:

$$x_1=2+3t,\ x_2=1-t,\ x_3=-1+2t,\ x_4=3-2t$$

and

$$x_1=7t,\ x_2=1,\ x_3=1+t,\ x_4=-1+2t.$$

Find the points of intersection of this straight line with the given straight lines.

886. Prove that any two straight lines in n-dimensional space can be embedded in a three-dimensional linear manifold.

887. Investigate, in general form, the condition for solvability of Problem 885 for two straight lines in n-dimensional space.

888. Prove that any two planes in n-dimensional space can be embedded in a five-dimensional linear manifold.

889. Give a description of all possible cases of the mutual location of two planes in n-dimensional space.

890. Prove that a linear manifold can be characterized as a set of vectors containing the linear combinations $\alpha X_1+(1-\alpha) X_2$ of any two vectors X_1, X_2 for arbitrary α.

Sec. 2. Elementary Geometry of n-Dimensional Euclidean Space

891. Determine the scalar product of the vectors X and Y:

(a) $X=(2, 1, -1, 2)$, $Y=(3, -1, -2, 1)$;

(b) $X=(1, 2, 1, -1)$, $Y=(-2, 3, -5, -1)$.

892. Determine the angle between the vectors X and Y:

(a) $X=(2, 1, 3, 2)$, $Y=(1, 2, -2, 1)$;

(b) $X=(1, 2, 2, 3)$, $Y=(3, 1, 5, 1)$;

(c) $X=(1, 1, 1, 2)$, $Y=(3, 1, -1, 0)$.

893. Determine the cosines of the angles between the straight line $x_1=x_2=\ldots=x_n$ and the axes of coordinates.

894. Determine the cosines of the interior angles of a triangle ABC which is specified by the coordinates of the vertices:

$A=(1, 2, 1, 2)$, $B=(3, 1, -1, 0)$, $C=(1, 1, 0, 1)$.

895. Find the lengths of the diagonals of an n-dimensional cube with side unity.

896. Find the number of diagonals of an n-dimensional cube which are orthogonal to a given diagonal.

897. In n-dimensional space, find n points with nonnegative coordinates such that the distances between the points and from the origin are unity. Place the first of these points on the first axis, the second, in the plane spanned by the first two axes, etc. Together with the coordinate origin, these points form the vertices of a regular simplex with unit edge.

898. Determine the coordinates of the centre and radius of a sphere circumscribed about the simplex of Problem 897.

899. Normalize the vector (3, 1, 2, 1).

900. Find the normalized vector orthogonal to the vectors (1, 1, 1, 1); (1, −1, −1, 1); (2, 1, 1, 3).

901. Construct an orthonormal basis of a space, taking for two vectors of this basis the vectors

$$\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) \text{ and } \left(\frac{1}{6}, \frac{1}{6}, \frac{1}{2}, -\frac{5}{6}\right).$$

902. By means of the orthogonalization process, find the orthogonal basis of a space generated by the vectors (1, 2, 1, 3); (4, 1, 1, 1); (3, 1, 1, 0).

903. Adjoin to the matrix

$$\begin{pmatrix} 1 & 1 & 1 & 2 & 1 \\ 1 & 0 & 0 & 1 & -2 \\ 2 & 1 & -1 & 0 & 2 \end{pmatrix}$$

two mutually orthogonal rows that are orthogonal to the first three rows.

904. Interpret the system of homogeneous linear equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= 0, \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= 0 \end{aligned}$$

and its fundamental system of solutions in a space of n dimensions, taking the coefficients of each equation for the coordinates of a vector.

905. Find an orthogonal and normalized fundamental system of solutions for the system of equations

$$3x_1 - x_2 - x_3 + x_4 = 0,$$
$$x_1 + 2x_2 - x_3 - x_4 = 0.$$

906. Decompose the vector X into a sum of two vectors, one of which lies in a space spanned by the vectors $A_1, A_2, \ldots, A_m$ and the other is orthogonal to this space (the orthogonal projection and the orthogonal component of the vector X):

(a) $X=(5, 2, -2, 2)$, $A_1=(2, 1, 1, -1)$,
$A_2=(1, 1, 3, 0)$;

(b) $X=(-3, 5, 9, 3)$, $A_1=(1, 1, 1, 1)$,
$A_2=(2, -1, 1, 1)$, $A_3=(2, -7, -1, -1)$.

907. Assuming the vectors $A_1, A_2, \ldots, A_m$ to be linearly independent, give formulas for computing the lengths of the components of the vector in Problem 906 when posed in general form.

908. Prove that of all vectors of a given space P, the smallest angle with a given vector X is formed by the orthogonal projection of the vector X on the space P.

909. Find the smallest angle between the vectors of the space P (spanned by the vectors $A_1, \ldots, A_m$) and the vector X:

(a) $X=(1, 3, -1, 3)$, $A_1=(1, -1, 1, 1)$,
$A_2=(5, 1, -3, 3)$;

(b) $X=(2, 2, -1, 1)$, $A_1=(1, -1, 1, 1)$,
$A_2=(-1, 2, 3, 1)$, $A_3=(1, 0, 5, 3)$.

910. Find the smallest angle formed by the vector $(1, 1, \ldots, 1)$ with the vectors of some m-dimensional coordinate space.

911. Prove that of all vectors $X-Y$, where X is a given vector and Y runs through a given space P, the vector $X-X'$, where X' is the orthogonal projection of X on P, is of smallest length. (This smallest length is called the distance from the point X to the space P.)

912. Determine the distance from the point X to the linear manifold $A_0 + t_1 A_1 + \ldots + t_m A_m$:

(a) $X=(1, 2, -1, 1)$, $A_0=(0, -1, 1, 1)$,
$A_1=(0, -3, -1, 5)$, $A_2=(4, -1, -3, 3)$;

(b) $X=(0, 0, 0, 0)$, $A_0=(1, 1, 1, 1)$, $A_1=(1, 2, 3, 4)$.

913. Consider a space of polynomials of degree not exceeding n. The scalar product of polynomials f_1, f_2 is defined as $\int_0^1 f_1(x) f_2(x)\, dx$. Find the distance from the origin to the linear manifold consisting of the polynomials $x^n + a_1 x^{n-1} + \ldots + a_n$.

914. Indicate a method for determining the shortest distance between the points of the two linear manifolds $X_0 + P$ and $Y_0 + Q$.

915. The vertices of a regular n-dimensional simplex (see Problem 897), the length of an edge of which is unity, are partitioned into two sets of $m+1$ and $n-m$ vertices. Linear manifolds of smallest dimension are passed through these sets of vertices. Determine the shortest distance between the points of these manifolds and determine the points for which it is realized.

***916.** Given, in a four-dimensional space, two planes spanned by the vectors A_1, A_2 and B_1, B_2. Find the smallest of the angles formed by the vectors of the first plane with the vectors of the second plane:

(a) $A_1 = (1, 0, 0, 0)$, $A_2 = (0, 1, 0, 0)$, $B_1 = (1, 1, 1, 1)$,

$B_2 = (2, -2, 5, 2)$;

(b) $A_1 = (1, 0, 0, 0)$, $A_2 = (0, 1, 0, 0)$, $B_1 = (1, 1, 1, 1)$,

$B_2 = (1, -1, 1, -1)$.

***917.** A four-dimensional cube is cut by a three-dimensional "plane" passing through the centre of the cube and orthogonal to a diagonal. Determine the shape of the solid obtained in the intersection.

***918.** Given a system of linearly independent vectors B_1, B_2, $\ldots$, B_m. The set of points made up of the endpoints of the vectors $t_1B_1 + t_2B_2 + \ldots + t_mB_m$, $0 \leqslant t_1 \leqslant 1$, $\ldots$, $0 \leqslant t_m \leqslant 1$, is called a parallelepiped constructed on the vectors B_1, B_2, $\ldots$, B_m. Determine the volume of the parallelepiped inductively as the volume of the "base" $[B_1, B_2, \ldots, B_{m-1}]$ multiplied by the "altitude" equal to the distance from the endpoint of vector B_m to the space spanned by the base. The "volume" of the one-dimensional "parallelepiped" $[B_1]$ is considered equal to the length of the vector B_1.

(a) Develop a formula for computing the square of the volume and assure yourself that the volume does not depend on the numbering of the vertices.

(b) Prove that $V[cB_1, B_2, \ldots, B_m] = |c| \cdot V[B_1, B_2, \ldots, B_m]$.

(c) Prove that $V[B_1'+B_1'', B_2, \ldots, B_m] \leqslant V[B_1', B_2, \ldots, B_m] + V[B_1'', B_2, \ldots, B_m]$ and determine when the equal sign holds true.

919. Prove that the volume of an n-dimensional parallelepiped in n-dimensional space is equal to the absolute value of the determinant made up of the coordinates of the generating vectors.

***920.** Let $C_1, C_2, \ldots, C_m$ be the orthogonal projections of the vectors $B_1, B_2, \ldots, B_m$ on some space. Prove that

$$V[C_1, C_2, \ldots, C_m] \leqslant V[B_1, B_2, \ldots, B_m].$$

***921.** Prove that

$$V[A_1, A_2, \ldots, A_m, B_1, \ldots, B_k] \leqslant V[A_1, \ldots, A_m] \cdot V[B_1, \ldots, B_k]$$

(cf. Problem 518).

922. Prove that

$$V[A_1, A_2, \ldots, A_m] \leqslant |A_1| \cdot |A_2| \ldots |A_m|$$

(cf. Problem 519).

923. Find the volume of an n-dimensional sphere using Cavalieri's principle.

924. Consider the space of polynomials whose degree does not exceed n. For the scalar product we take $\int\limits_0^1 f_1(x) f_2(x)\, dx$. Find the volume of the parallelepiped formed by the vectors of the basis relative to which the coefficients of the polynomial are its coordinates.

Sec. 3. Eigenvalues and Eigenvectors of a Matrix

925. Find the eigenvalues and eigenvectors of the following matrices:

(a) $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, (b) $\begin{pmatrix} 3 & 4 \\ 5 & 2 \end{pmatrix}$, (c) $\begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}$,

(d) $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$, (e) $\begin{pmatrix} 5 & 6 & -3 \\ -1 & 0 & 1 \\ 1 & 2 & -1 \end{pmatrix}$,

(f) $\begin{pmatrix} 2 & -1 & 2 \\ 5 & -3 & 3 \\ -1 & 0 & -2 \end{pmatrix}$, (g) $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, (h) $\begin{pmatrix} 0 & 2 & 1 \\ -2 & 0 & 3 \\ -1 & -3 & 0 \end{pmatrix}$,

(i) $\begin{pmatrix} 3 & 1 & 0 \\ -4 & -1 & 0 \\ 4 & -8 & -2 \end{pmatrix}$, (j) $\begin{pmatrix} 2 & 5 & -6 \\ 4 & 6 & -9 \\ 3 & 6 & -8 \end{pmatrix}$.

926. Knowing the eigenvalues of the matrix A, find the eigenvalues of the matrix A^{-1}.

927. Knowing the eigenvalues of the matrix A, find the eigenvalues of the matrix A^2.

928. Knowing the eigenvalues of the matrix A, find the eigenvalues of the matrix A^m.

929. Knowing the characteristic polynomial $F(\lambda)$ of the matrix A (of order n), find the determinant of the matrix $f(A)$, where $f(x)=b_0(x-\xi_1)(x-\xi_2)\dots(x-\xi_m)$.

930. Knowing the eigenvalues of the matrix A, find the determinant of the matrix $f(A)$, where $f(x)$ is a polynomial.

931. Knowing the eigenvalues of the matrix A, find the eigenvalues of the matrix $f(A)$.

932. Prove that all the eigenvectors of the matrix A are eigenvectors of the matrix $f(A)$.

***933.** Find the eigenvalues of the matrix

$$\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \varepsilon & \varepsilon^2 & \dots & \varepsilon^{n-1} \\ 1 & \varepsilon^2 & \varepsilon^4 & \dots & \varepsilon^{2(n-1)} \\ \dots & \dots & \dots & \dots & \dots \\ 1 & \varepsilon^{(n-1)} & \varepsilon^{2(n-1)} & \dots & \varepsilon^{(n-1)^2} \end{pmatrix}$$

where $\varepsilon=\cos\frac{2\pi}{n}+i\sin\frac{2\pi}{n}$, n an odd number.

***934.** Find the sum

$$1+\varepsilon+\varepsilon^4+\dots+\varepsilon^{(n-1)^2}.$$

935. Find the eigenvalues of the matrices:

(a) $\begin{pmatrix} 0 & x & x & \dots & x \\ y & 0 & x & \dots & x \\ y & y & 0 & \dots & x \\ \dots & \dots & \dots & \dots & \dots \\ y & y & y & \dots & 0 \end{pmatrix}$, (b) $\begin{pmatrix} a_1 & a_2 & \dots & a_n \\ a_n & a_1 & \dots & a_{n-1} \\ \dots & \dots & \dots & \dots \\ a_2 & a_3 & \dots & a_1 \end{pmatrix}$,

(c) $\begin{pmatrix} 0 & 1 & & & & & \\ -1 & 0 & 1 & & & & \\ & -1 & 0 & & & & \\ & & & \cdot & & & \\ & & & & \cdot & & \\ & & & & & \cdot & \\ & & & & & 0 & 1 \\ & & & & & -1 & 0 \end{pmatrix}$.

***936.** Knowing the eigenvalues of the matrices A and B, find the eigenvalues of their Kronecker product.

937. Prove that the characteristic polynomials of the matrices AB and BA coincide for arbitrary square matrices A and B.

938. Prove that the characteristic polynomials of the matrices AB and BA differ solely in the factor $(-\lambda)^{n-m}$. Here, A is a rectangular matrix with m rows and n columns, and B is an n-by-m matrix, $n>m$.

Sec. 4. Quadratic Forms and Symmetric Matrices

939. Transform the following quadratic forms to a sum of squares:

(a) $x_1^2+2x_1x_2+2x_2^2+4x_2x_3+5x_3^2$,

(b) $x_1^2-4x_1x_2+2x_1x_3+4x_2^2+x_3^2$,

(c) $x_1x_2+x_2x_3+x_3x_1$,

(d) $x_1^2-2x_1x_2+2x_1x_3-2x_1x_4+x_2^2+2x_2x_3-4x_2x_4+x_3^2-2x_4^2$,

(e) $x_1^2+x_1x_2+x_3x_4$.

940. Transform the quadratic form

$$\sum_{i=1}^{n} x_i^2+\sum_{i<k} x_i x_k$$

to diagonal form.

941. Transform the quadratic form

$$\sum_{i<k} x_i x_k$$

to diagonal form.

942. Prove that all the principal minors of the positive quadratic form are positive.

***943.** Let the quadratic form

$$\begin{aligned} f = a_{11} x_1^2 + a_{12} x_1 x_2 + \ldots + a_{1n} x_1 x_n \\ + a_{21} x_2 x_1 + a_{22} x_2^2 + \ldots + a_{2n} x_2 x_n \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ + a_{n1} x_n x_1 + a_{n2} x_n x_2 + \ldots + a_{nn} x_n^2 \end{aligned}$$

be reducible to the diagonal form $\alpha_1 x_1'^2 + \alpha_2 x_2'^2 + \ldots + \alpha_n x_n'^2$ by the "triangular" transformation

$$\begin{aligned} x_1' &= x_1 + b_{12} x_2 + \ldots + b_{1n} x_n \\ x_2' &= \qquad\quad x_2 + \ldots + b_{2n} x_n \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ x_n' &= \qquad\qquad\qquad\quad x . \end{aligned}$$

It is required to:

(a) express the coefficients $\alpha_1, \alpha_2, \ldots, \alpha_n$ in terms of the coefficients a_{ik};

(b) express the discriminants of the forms $f_k(x_{k+1}, \ldots, x_n) = f - \alpha_1 x_1'^2 - \ldots - \alpha_k x_k'^2$ in terms of the coefficients a_{ik}.

Find the condition under which a triangular transformation of the indicated type is possible.

944. Prove that the necessary and sufficient condition for positivity of the quadratic form

$$\begin{aligned} f = a_{11} x_1^2 + a_{12} x_1 x_2 + \ldots + a_{1n} x_1 x_n \\ + a_{21} x_2 x_1 + a_{22} x_2^2 + \ldots + a_{2n} x_2 x_n \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ + a_{n1} x_n x_1 + a_{n2} x_n x_2 + \ldots + a_{nn} x_n^2 \end{aligned}$$

is fulfilment of the inequalities

$$a_{11} > 0; \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0; \quad \ldots; \quad \begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{vmatrix} > 0$$

(Sylvester's condition).

***945.** Prove that if to a positive quadratic form we add the square of a linear form, the discriminant of the former increases.

***946.** Let $f(x_1, x_2, \ldots, x_n) = a_{11} x_1^2 + \ldots$ be a positive quadratic form,

$$\varphi(x_2, \ldots, x_n) = f(0, x_2, \ldots, x_n)$$

D_f and D_φ their discriminants. Prove that

$$D_f \leqslant a_{11} D_\varphi.$$

947. Let

$$f(x_1, x_2, \dots, x_n) = l_1^2 + l_2^2 + \dots + l_p^2 - l_{p+1}^2 - l_{p+2}^2 - \dots - l_{p+q}^2$$

where $l_1, l_2, \dots, l_p, l_{p+1}, l_{p+2}, \dots, l_{p+q}$ are real linear forms in $x_1, x_2, \dots, x_n$. Prove that the number of positive squares in a canonical representation of the form f does not exceed p, and the number of negative squares does not exceed q.

***948.** Let $s_0, s_1, \dots$ be power sums of the roots of the equation $x^n + a_1 x^{n-1} + \dots + a_n = 0$ with real coefficients. Prove that the number of negative squares in a canonical representation of the quadratic form $\sum_{i,k=1}^{n} s_{i+k+2} x_i x_k$ is equal to the number of pairs of conjugate complex roots of the given equation.

Prove the following theorems:

949. Fulfillment of the following inequalities is a necessary and sufficient condition for all the roots of an equation with real coefficients to be real and distinct:

$$\begin{vmatrix} s_0 & s_1 \\ s_1 & s_2 \end{vmatrix} > 0; \quad \begin{vmatrix} s_0 & s_1 & s_2 \\ s_1 & s_2 & s_3 \\ s_2 & s_3 & s_4 \end{vmatrix} > 0; \quad \dots; \quad \Delta = \begin{vmatrix} s_0 & s_1 & \dots & s_{n-1} \\ s_1 & s_2 & \dots & s_n \\ \dots & \dots & \dots & \dots \\ s_{n-1} & s_n & \dots & s_{2n-2} \end{vmatrix} > 0.$$

***950.** If the quadratic forms

$$\begin{aligned} f = a_{11} x_1^2 + a_{12} x_1 x_2 + \dots + a_{1n} x_1 x_n \\ + a_{21} x_2 x_1 + a_{22} x_2^2 + \dots + a_{2n} x_2 x_n \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ + a_{n1} x_n x_1 + a_{n2} x_n x_2 + \dots + a_{nn} x_n^2 \end{aligned}$$

and

$$\begin{aligned} \varphi = b_{11} x_1^2 + b_{12} x_1 x_2 + \dots + b_{1n} x_1 x_n \\ + b_{21} x_2 x_1 + b_{22} x_2^2 + \dots + b_{2n} x_2 x_n \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ + b_{n1} x_n x_1 + b_{n2} x_n x_2 + \dots + b_{nn} x_n^2 \end{aligned}$$

are nonnegative, then the form

$$\begin{aligned}(f,\ \varphi)=a_{11}b_{11}x_1^2+a_{12}b_{12}x_1x_2+\ldots+a_{1n}b_{1n}x_1x_n\\ +a_{21}b_{21}x_2x_1+a_{22}b_{22}x_2^2+\ldots+a_{2n}b_{2n}x_2x_n\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ +a_{n1}b_{n1}x_nx_1+a_{n2}b_{n2}x_nx_2+\ldots+a_{nn}b_{nn}x_n^2\end{aligned}$$

is nonnegative.

951. Transform the following quadratic forms to canonical form by an orthogonal transformation:

(a) $2x_1^2+x_2^2-4x_1x_2-4x_2x_3$,

(b) $x_1^2+2x_2^2+3x_3^2-4x_1x_2-4x_2x_3$,

(c) $3x_1^2+4x_2^2+5x_3^2+4x_1x_2-4x_2x_3$,

(d) $2x_1^2+5x_2^2+5x_3^2+4x_1x_2-4x_1x_3-8x_2x_3$,

(e) $x_1^2-2x_2^2-2x_3^2-4x_1x_2+4x_1x_3+8x_2x_3$,

(f) $5x_1^2+6x_2^2+4x_3^2-4x_1x_2-4x_1x_3$,

(g) $3x_1^2+6x_2^2+3x_3^2-4x_1x_2-8x_1x_3-4x_2x_3$,

(h) $7x_1^2+5x_2^2+3x_3^2-8x_1x_2+8x_2x_3$,

(i) $2x_1^2+2x_2^2+2x_3^2+2x_4^2-4x_1x_2+2x_1x_4+2x_2x_3-4x_3x_4$,

(j) $2x_1x_2+2x_3x_4$,

(k) $x_1^2+x_2^2+x_3^2+x_4^2+2x_1x_2-2x_1x_4-2x_2x_3+2x_3x_4$,

(l) $2x_1x_2+2x_1x_3-2x_1x_4-2x_2x_3+2x_2x_4+2x_3x_4$,

(m) $x_1^2+x_2^2+x_3^2+x_4^2-2x_1x_2+6x_1x_3-4x_1x_4$
$-4x_2x_3+6x_2x_4-2x_3x_4$,

(n) $8x_1x_3+2x_1x_4+2x_2x_3+8x_2x_4$.

952. Transform the following quadratic forms to canonical form by an orthogonal transformation:

(a) $\sum\limits_{i=1}^{n} x_i^2+\sum\limits_{i<k} x_ix_k$, (b) $\sum\limits_{i<k} x_ix_k$.

953. Transform the form

$$x_1x_2+x_2x_3+\ldots+x_{n-1}x_n$$

to canonical form by an orthogonal transformation.

954. Prove that if all the eigenvalues of a real symmetric matrix A lie in the interval $[a, b]$, then the quadratic form with matrix $A-\lambda E$ is negative for $\lambda>b$ and positive for $\lambda<a$. The converse holds true as well.

955. Prove that if all the eigenvalues of a real symmetric matrix A lie in the interval $[a, c]$ and all the eigenvalues of a real symmetric matrix B lie in the interval $[b, d]$ then all the eigenvalues of the matrix $A+B$ lie in the interval $[a+b, c+d]$.

956. Let us call the positive square root of the largest eigenvalues of the matrix $\tilde{A}A$ (A is a real square matrix, $\tilde{A}$ is its transpose) the norm of the matrix A and denote it by $\| A \|$. Prove that

(a) $\| A \| = \| \tilde{A} \|$,

(b) $| AX | \leqslant \| A \| \cdot | X |$; the equality holds for some vector X_0,

(c) $\| A+B \| \leqslant \| A \| + \| B \|$,

(d) $\| AB \| \leqslant \| A \| \cdot \| B \|$,

(e) the moduli of all eigenvalues of the matrix A do not exceed $\| A \|$.

957. Prove that any real nonsingular matrix can be represented as a product of an orthogonal matrix and a triangular matrix of the form

$$\begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ & b_{22} & \dots & b_{2n} \\ & & \ddots & \vdots \\ & & & b_{nn} \end{pmatrix}$$

with positive diagonal elements b_{ii} and that this representation is unique.

958. Prove that any real nonsingular matrix is representable in the form of a product of an orthogonal matrix and a symmetric matrix corresponding to some positive quadratic form.

959. Let there be a quadric surface in n-dimensional space given by the equation

$$\begin{aligned} a_{11} x_1^2 + a_{12} x_1 x_2 + \dots + a_{1n} x_1 x_n & \\ + a_{21} x_2 x_1 + a_{22} x_2^2 + \dots + a_{2n} x_2 x_n & \\ \dots\dots\dots\dots\dots\dots\dots\dots & \\ + a_{n1} x_n x_1 + a_{n2} x_n x_2 + \dots + a_{nn} x_n^2 & \\ + 2b_1 x_1 + 2b_2 x_2 + \dots + 2b_n x_n + c = 0 & \end{aligned}$$

or, in abbreviated notation, $AX \cdot X + 2B \cdot X + c = 0$. Prove that for the centre of the surface to exist it is necessary and sufficient that the rank of the matrix A be equal to the rank of the matrix (A, B).

960. Prove that the equation of a central quadric surface may be reduced to canonical form

$$\alpha_1 x_1^2 + \ldots + \alpha_r x_r^2 + \gamma = 0$$

by a translati on of the origin and by an orthogonal transformation.

961. Prove that the equation of a noncentral quadric surface may be reduced to canonical form

$$\alpha_1 x_1^2 + \ldots + \alpha_r x_r^2 = 2x_{r+1}$$

by a translation of the origin and by an orthogonal transformation.

Sec. 5. Linear Transformations. Jordan Canonical Form

962. Establish that the dimension of a subspace into which the entire space is mapped under a linear transformation is equal to the rank of the matrix of this linear transformation.

963. Let Q be a subspace of dimension q of the space R of dimension n, and let Q' be the image of Q under a linear transformation of rank r of the space R. Prove that the dimension q' of space Q' satisfies the inequalities

$$q + r - n \leqslant q' \leqslant \min(q, r).$$

964. Using the result of Problem 963, establish that the rank ρ of the product of two matrices of ranks r_1 and r_2 satisfies the inequalities

$$r_1 + r_2 - n \leqslant \rho \leqslant \min(r_1, r_2).$$

***965.** Let P and Q be any complementary subspaces of the space R. Then any vector $X \in R$ decomposes uniquely into a sum of the vectors $Y \in P$ and $Z \in Q$. The transformation consisting in going from vector X to its component Y is called projection on P parallel to Q. Prove that projection is a linear transformation and its matrix A (in any basis) satisfies the condition $A^2 = A$. Conversely, any linear transformation whose matrix satisfies the condition $A^2 = A$ is a projection.

*__966.__ The projection is termed orthogonal if $P \perp Q$. Prove that in any orthonormal basis, the matrix of orthogonal projection is symmetric. Conversely, any symmetric idempotent matrix of the same degree is a matrix of orthogonal projection.

*__967.__ Prove that all nonzero eigenvalues of a skew-symmetric matrix are pure imaginaries, and the real and imaginary parts of the corresponding eigenvectors are equal in length and orthogonal.

*__968.__ Prove that for a skew-symmetric matrix A it is possible to find an orthogonal matrix P such that

$$P^{-1}AP=\begin{pmatrix} 0 & a_1 & & & & & & & & & \\ -a_1 & 0 & & & & & & & & & \\ & & 0 & a_2 & & & & & & & \\ & & -a_2 & 0 & & & & & & & \\ & & & & \ddots & & & & & & \\ & & & & & 0 & a_k & & & & \\ & & & & & -a_k & 0 & & & & \\ & & & & & & & 0 & & & \\ & & & & & & & & \ddots & & \\ & & & & & & & & & & 0 \end{pmatrix}$$

(all elements not indicated are zero; $a_1, a_2, \ldots, a_k$ are real numbers).

__969.__ Prove the theorem: if A is a skew-symmetric matrix, then the matrix $(E-A)\,(E+A)^{-1}$ is an orthogonal matrix without -1 as eigenvalue. Conversely, every orthogonal matrix that does not have -1 as an eigenvalue can be represented in this form.

*__970.__ Prove that the moduli of all eigenvalues of an orthogonal matrix are equal to 1.

*__971.__ Prove that eigenvectors of an orthogonal matrix which belong to a complex eigenvalue are of the form $X+iY$, where X, Y are real vectors equal in length and orthogonal.

*__972.__ Prove that every orthogonal matrix can be represented as

$$Q^{-1}TQ$$

where Q is an orthogonal matrix and T is of the form

$$\begin{pmatrix} \cos\varphi_1 & -\sin\varphi_1 & & & & & & & & & & & & & & \\ \sin\varphi_1 & \cos\varphi_1 & & & & & & & & & & & & & & \\ & & \cos\varphi_2 & -\sin\varphi_2 & & & & & & & & & & & & \\ & & \sin\varphi_2 & \cos\varphi_2 & & & & & & & & & & & & \\ & & & & \cdot & & & & & & & & & & & \\ & & & & & \cdot & & & & & & & & & & \\ & & & & & & \cdot & & & & & & & & & \\ & & & & & & & 1 & & & & & & & & \\ & & & & & & & & 1 & & & & & & & \\ & & & & & & & & & \cdot & & & & & & \\ & & & & & & & & & & \cdot & & & & & \\ & & & & & & & & & & & \cdot & & & & \\ & & & & & & & & & & & & 1 & & & \\ & & & & & & & & & & & & & -1 & & \\ & & & & & & & & & & & & & & \cdot & \\ & & & & & & & & & & & & & & & \ddots \\ & & & & & & & & & & & & & & & & -1 \\ & & & & & & & & & & & & & & & & & -1 \end{pmatrix}$$

(all other elements being equal to zero).

973. Reduce the following matrices to the Jordan normal form:

(a) $\begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 0 \\ -2 & -2 & -1 \end{pmatrix}$, (b) $\begin{pmatrix} 4 & 6 & 0 \\ -3 & -5 & 0 \\ -3 & -6 & 1 \end{pmatrix}$,

(c) $\begin{pmatrix} 13 & 16 & 16 \\ -5 & -7 & -6 \\ -6 & -8 & -7 \end{pmatrix}$, (d) $\begin{pmatrix} 3 & 0 & 8 \\ 3 & -1 & 6 \\ -2 & 0 & -5 \end{pmatrix}$,

(e) $\begin{pmatrix} -4 & 2 & 10 \\ -4 & 3 & 7 \\ -3 & 1 & 7 \end{pmatrix}$, (f) $\begin{pmatrix} 7 & -12 & -2 \\ 3 & -4 & 0 \\ -2 & 0 & -2 \end{pmatrix}$,

(g) $\begin{pmatrix} -2 & 8 & 6 \\ -4 & 10 & 6 \\ 4 & -8 & -4 \end{pmatrix}$, (h) $\begin{pmatrix} 0 & 3 & 3 \\ -1 & 8 & 6 \\ 2 & -14 & -10 \end{pmatrix}$,

(i) $\begin{pmatrix} -1 & 1 & 1 \\ -5 & 21 & 17 \\ 6 & -26 & -21 \end{pmatrix}$, (j) $\begin{pmatrix} 8 & 30 & -14 \\ -6 & -19 & 9 \\ -6 & -23 & 11 \end{pmatrix}$,

(k) $\begin{pmatrix} 4 & 5 & -2 \\ -2 & -2 & 1 \\ -1 & -1 & 1 \end{pmatrix}$, (l) $\begin{pmatrix} 3 & 7 & -3 \\ -2 & -5 & 2 \\ -4 & -10 & 3 \end{pmatrix}$,

(m) $\begin{pmatrix} 9 & 22 & -6 \\ -1 & -4 & 1 \\ 8 & 16 & -5 \end{pmatrix}$, (n) $\begin{pmatrix} 1 & -1 & 2 \\ 3 & -3 & 6 \\ 2 & -2 & 4 \end{pmatrix}$,

(o) $\begin{pmatrix} 1 & 1 & -1 \\ -3 & -3 & 3 \\ -2 & -2 & 2 \end{pmatrix}$.

974. Reduce the following matrices to the Jordan normal form:

(a) $\begin{pmatrix} 3 & 1 & 0 & 0 \\ -4 & -1 & 0 & 0 \\ 7 & 1 & 2 & 1 \\ -17 & -6 & -1 & 0 \end{pmatrix}$, (b) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$,

(c) $\begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}$.

***975.** Prove that any periodic matrix A (satisfying the condition $A^m = E$ for some natural m) is reducible to the diagonal canonical form.

***976.** Knowing the eigenvalues of the matrix A, find the eigenvalues of the matrix A'_m composed of appropriately arranged mth-order minors of the matrix A (see Problem 531).

977. Prove that any matrix A can be transformed into its transpose.

***978.** Prove that any matrix can be represented as a product of two symmetric matrices, one of which is nonsingular.

979. Starting with a given matrix A of order n, construct a sequence of matrices via the following process:

$$\begin{array}{lll} A_1 = A, & \operatorname{tr} A_1 = p_1, & A_1 - p_1 E = B_1, \\ B_1 A = A_2, & \frac{1}{2} \operatorname{tr} A_2 = p_2, & A_2 - p_2 E = B_2, \\ B_2 A = A_3, & \frac{1}{3} \operatorname{tr} A_3 = p_3, & A_3 - p_3 E = B_3, \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ B_{n-1} A = A_n, & \frac{1}{n} \operatorname{tr} A_n = p_n, & A_n - p_n E = B_n \end{array}$$

where tr A_i is the trace of matrix A_i (the sum of the diagonal elements). Prove that $p_1, p_2, \ldots, p_n$ are the coefficients of the characteristic polynomial of the matrix A written in the form $(-1)^n [\lambda^n - p_1 \lambda^{n-1} - p_2 \lambda^{n-2} - \ldots - p_n]$; matrix B_n is a zero matrix; finally, if A is nonsingular, then $\frac{1}{p_n} B_{n-1} = A^{-1}$.

***980.** For the equation $XY - YX = C$ to be solvable in terms of square matrices X, Y, it is necessary and sufficient that the trace of the matrix C be zero. Prove this.

PART II. HINTS TO SOLUTIONS

CHAPTER 1
COMPLEX NUMBERS

11. See Problem 10.

13. Demonstrate the validity of the theorem for each of the four operations on the two numbers and take advantage of the method of mathematical induction.

18. Use the fact that the left members are easily represented as a sum of two squares.

27. Set $x=a+bi$, $y=c+di$.

28. Set $z=\cos\varphi+i\sin\varphi$.

31. Set $z=t^2$, $z'=t'^2$. Use Problem 27.

37. Go over to the trigonometric form.

38. $1+\omega=-\omega^2$.

40. Pass to the half-angle.

41. Convince yourself that $z=\cos\theta\pm i\sin\theta$; $\frac{1}{z}=\cos\theta\mp i\sin\theta$. Take advantage of De Moivre's formula.

51. Set $\alpha=\cos x+i\sin x$. Then $\cos^{2m}x=\left(\frac{\alpha+\alpha^{-1}}{2}\right)^{2m}$, etc.

52. Show that the coefficient of $(2\cos x)^{m-2p}$ is equal to $(-1)^p(C^p_{m-p}+C^{p-1}_{m-p-1})$. Take advantage of the method of mathematical induction.

53. This is similar to Problem 52.

54. Make use of the binomial expansion of $(1+i)^n$.

55. Use Problem 54.

56. Expand $\left(1+i\frac{\sqrt{3}}{3}\right)^n$ using Newton's binomial formula.

68. Show that the problem reduces to computing the limit of the sum $1+\alpha+\alpha^2+\ldots$, where $\alpha=\frac{-1+i}{2}$.

69. Take advantage of the fact that $\sin^2\alpha=\frac{1}{2}-\frac{\cos 2\alpha}{2}$.

71. Use the fact that

$$\cos^3\alpha=\frac{\cos 3\alpha}{4}+\frac{3\cos\alpha}{4},\quad \sin^3\alpha=\frac{3\sin\alpha}{4}-\frac{\sin 3\alpha}{4}.$$

72. In computing sums of the type $1+2a+3a^2+\ldots+na^{n-1}$ and $1+2^2a+3^2a^2+\ldots+n^2a^{n-1}$ it is useful first to multiply them by $1-a$.

76. $x_1=\alpha+\beta$, $x_2=\alpha\omega+\beta\omega^2$, $x_3=\alpha\omega^3+\beta\omega$, $\alpha^3+\beta^3=-q$, $3\alpha\beta=-p$.

77. Multiply by -27 and regard the left member as the discriminant of some cubic equation.

78. Set $x=\alpha+\beta$.

87. Show that $\varepsilon^n=-1$.

88. If $\varepsilon=\cos\frac{2\pi}{n}+i\sin\frac{2\pi}{n}$, then the desired sum can be written as $1+\varepsilon+\varepsilon^2+\ldots+\varepsilon^{n-1}$.

89. Consider two cases: (1) k is divisible by n; (2) k is not divisible by n.

91. 92. Multiply by $1-\varepsilon$.

94. (a) Subtract from the sum of all 15th roots of 1 the sum of the roots belonging to the exponents 1, 3, and 5.

97. The length of a side of a regular 14-sided polygon of radius unity is equal to $2\sin\frac{\pi}{14}$. Use the fact that $\cos\frac{4\pi}{7}+i\sin\frac{4\pi}{7}$ satisfies the equation $x^6+x^5+x^4+x^3+x^2+x+1=0$.

98. (1) If $x_1, x_2, \ldots, x_n$ are roots of the equation $a_0x^n+a_1x^{n-1}+\ldots+a_n=0$, then $a_0x^n+a_1x^{n-1}+\ldots+a_n=a_0(x-x_1)\ldots(x-x_n)$.

(2) If ε is an nth root of 1, then $\bar{\varepsilon}$, the conjugate of ε, is also an nth root of 1.

99. In the identities obtained from Problem 98 set $x=1$.

100. Take advantage of the factorization of x^n-1 into linear factors.

101. In the factorization of x^n-1 into linear factors set: (1) $x=\cos\theta+i\sin\theta$, (2) $x=\cos\theta-i\sin\theta$.

103. Take advantage of the fact that the moduli of conjugate complex numbers are equal.

105. (a) Reduce the equation to the form $\left(\frac{x+1}{x-1}\right)^m=1$.

107. Let

$$S=\cos\varphi+C_n^1\cos(\varphi+\alpha)x+\ldots+\cos(\varphi+n\alpha)x^n,$$

$$T=\sin\varphi+C_n^1\sin(\varphi+\alpha)x+\ldots+\sin(\varphi+n\alpha)x^n.$$

Compute $S+Ti$ and $S-Ti$ and determine S from the resulting equations.

113. First prove that $\varphi(p^\alpha)=p^\alpha\left(1-\frac{1}{p}\right)$, if p is a prime number. To do this, count the numbers not exceeding p^α that are divisible by p.

116. Prove that all roots of $x^{p^{m-1}}-1$ and only such roots are not primitive roots of $x^{p^m}-1$.

117. Show that if n is odd, then to obtain all the primitive roots of degree $2n$ of unity it is sufficient to multiply all primitive nth roots by -1.

119. Use Problem 118.

120. Use Problems 115, 116, 111 and show that (1) $\mu(p)=-1$ if p is prime; (2) that $\mu(p^\alpha)=0$ if p is prime, $\alpha>1$, (3) $\mu(ab)=\mu(a)\mu(b)$ if a and b are relative prime.

122. Show that if $\varepsilon_k=\cos\frac{2k\pi}{n}+i\sin\frac{2k\pi}{n}$ belongs to the exponent n_1, then $x-\varepsilon_k$ will enter the right member of the equation being proved to the power $\Sigma\mu(d_1)$, where d_1 runs through all divisors $\frac{n}{n_1}$.

123. Consider the cases: (1) n is the power of a prim. (2) n is the product of powers of distinct primes. For Case (1) use Problem 11 for (2) use Problems 119 and 122.

124. Consider the cases: (1) n is odd and exceeds 1; (2) $n=2^k$; (3) $n=2n_1$, n_1 is odd and exceeds 1; (4) $n=2^k n_1$, where $k>1$, n_1 is odd and exceeds 1.

125. Use the identity

$$x_1 x_2+x_1 x_3+\ldots+x_{n-1} x_n$$
$$=\frac{(x_1+x_2+\ldots+x_n)^2-(x^1+x_2+\ldots+x_n^2)}{2}.$$

Consider the cases: (1) n is odd; (2) $n=2n_1$, n_1 is odd; (3) $n=2^k n_1$, where $k>1$, n_1 is odd.

126. Multiply the sum S by its conjugate and take into account that ε^{x^2} does not change when $x+n$ is substituted for x.

CHAPTER 2

EVALUATION OF DETERMINANTS

132. Bear in mind that each pair of elements of a permutation constitutes an inversion.

133. The number of inversions in the second permutation is equal to the number of orders in the first.

145. Show that each term has 0 for a factor.

149, 150. Replace rows by columns.

153. Find out how the determinant will change if its columns are permuted in some fashion.

154. (a) Note that when $x=a_i$, the determinant has two identical rows.

155. To the last column add the first multiplied by 100 and the second multiplied by 10.

156. First subtract the first column from each column.

163. Subtract the first column from the second.

179. Add the first row to all other rows.

180-182. Subtract the first row from all other rows.

183. Subtract the second row from all other rows.

184. Add the first row to the second.

185. Add all columns to the first.

186, 187. From the first column subtract the second, add the third, etc.

188. Expand by elements of the first column or add to the last row the first multiplied by x^n, the second multiplied by x^{n-1}, etc.

189. Add to the last column the first multiplied by x^{n-1}, the second multiplied by x^{n-2}, etc.

190. Construct a determinant equal to $f(x+1)-f(x)$. In the resulting determinant subtract from the last column the first, the second multiplied by x, the third multiplied by x^2, etc.

191. Multiply the last column by $a_1, a_2, \ldots, a_n$ and subtract, respectively, from the first, second, ..., nth column.

192. Add all columns to the first.

194. Add all columns to the last one.

195. Take a_1 out of the first column, a_2 out of the second, etc. Add to the last column all the preceding columns.

196. Take h out of the first column; add the first column to the second.

197. Multiply the first row and the first column by x.

198. Subtract from each row the first multiplied successively by $a_1, a_2, \ldots, a_n$. From each column subtract the first multiplied successively by $a_1, a_2, \ldots, a_n$.

199. Add all columns to the first.

200. Add to the first column all the others.

201. From each column, beginning with the last, subtract the preceding column multiplied by a.

202. From each row, beginning with the last, subtract the preceding row. Then to each column add the first.

203. Multiply the first row by b_0, the second by b_1, etc. To the first row add all succeeding rows.

204. Take a out of the first row and subtract the first row from the second.

205. Expand by elements of the first row.

206. Represent as a sum of two determinants.

208. Add a zero to each off-diagonal element and represent the determinant as a sum of 2^n determinants. Use Problem 206 or 207.

211. Multiply the first column by x^{n-1}, the second by x^{n-2}, etc.

212. Expand by elements of the last column and show that $\Delta_n = x_n \Delta_{n-1} + a_n x_1 x_2 \ldots x_{n-1}$ (Δ_n denotes a determinant of order n). Use mathematical induction in computing the determinant.

213. Expand by elements of the last column and show that $\Delta_{n+1} = x_n \Delta_n + a_n y_1 y_2 \ldots y_n$.

214. Take a_1 out of the second column, a_2 out of the third, ..., and a_n out of the $(n+1)$th. Reverse the sign of the first column and add all columns to the first.

215. Expand in terms of elements of the first row.

216. Expand in terms of elements of the first row and show that $\Delta_n = a_1 a_2 \ldots a_{n-1} - \Delta_{n-1}$.

219. Use the result of Problem 217.

221. Expand by elements of the first row and show that $\Delta_n = x\Delta_{n-1} - \Delta_{n-2}$.

222. From the last row subtract the second last multiplied by $\dfrac{y_n}{y_{n-1}}$.

Show that $\Delta_n = \dfrac{y_n}{y_{n-1}} (x_n y_{n-1} - x_{n-1} y_n) \Delta_{n-1}$.

223. Represent as a sum of two determinants and show that $\Delta_n = a_n \Delta_{n-1} + a_1 a_2 \ldots a_{n-1}$.

225. Represent as a sum of two determinants and show that

$$\Delta_n = (a_n - x)\Delta_{n-1} + x(a_1 - x) \ldots (a_{n-1} - x).$$

226. Setting $x_n = (x_n - a_n) + a_n$, represent the determinant in the form of a sum of two determinants and show that

$$\Delta_n = (x_n - a_n)\Delta_{n-1} + a_n (x_1 - a_1)(x_2 - a_2) \ldots (x_{n-1} - a_{n-1}).$$

227. Represent in the form of a sum of two determinants and show that $\Delta_n=(x_n-a_n b_n)\ \Delta_{n-1}+a_n b_n (x_1-a_1 b_1)\ \dots\ (x_{n-1}-a_{n-1} b_{n-1})$.

228. Represent in the form of a sum of two determinants and show that

$$\Delta_n=-m\,\Delta_{n-1}+(-1)^{n-1}m^{n-1}x_n.$$

230. Expand by elements of the first row and show that

$$\Delta_{2n}=(a^2-b^2)\,\Delta_{2n-2}.$$

231. From each row subtract the preceding one and add to the second all subsequent rows. Then, expanding the determinant by elements of the last row, show that

$$\Delta_n=[a+(n-1)\,b]\ \Delta_{n-1}+a\,(a+b)\ \dots\ [a+(n-2)\,b].$$

232. Represent as a sum of two determinants and show that

$$\Delta_n=x\,(x-2a_n)\,\Delta_{n-1}+a_n^2\,x^{n-1}\prod_{i=1}^{n-1}(x-2a_i).$$

233. Setting $(x-a_n)^2=x\,(x-2a_n)+a_n^2$, represent the determinant as a sum of two determinants and show that

$$\Delta_n=x\,(x-2a_n)\,\Delta_{n-1}+a_n^2\,x^{n-1}\,(x-2a_1)\ \dots\ (x-2a_{n-1}).$$

234. Represent as a sum of two determinants and show that

$$\Delta_n=\Delta_{n-1}+(-1)^n\,b_1 b_2\ \dots\ b_n.$$

235. Represent the last element of the last row as a_n-a_n. Prove that

$$\Delta_n=(-1)^{n-1}\,b_1 b_2 \dots b_{n-1}\,a_n-a_n\,\Delta_{n-1}.$$

236. From each row subtract the next.

237. Set $1=x+(1-x)$ in the upper left corner. Represent the determinant as a sum of two determinants. Use the result of Problem 236.

238. Multiply the second row by x^{n-1}, the third by x^{n-2}, ..., the nth by x. From the first column take out x^n, from the second, x^{n-1}, ..., from the nth, x.

239. Use the suggestion of the preceding problem.

240. From each column subtract the preceding one (begin with the last column). Then from each row subtract the preceding one. Prove that $\Delta_n=\Delta_{n-1}$. When calculating, bear in mind that $C_n^k=C_{n-1}^k+C_{n-1}^{k-1}$.

241. From each column subtract the preceding one.

242. From each row subtract the preceding one. Prove that $\Delta_n=\Delta_{n-1}$.

243. Take m out of the first row, $m+1$ out of the second, ..., $m+n$ out of the last. Take $\frac{1}{k}$ out of the first column, $\frac{1}{k+1}$ out of the second, etc. Repeat this operation until all elements of the first column become equal to 1.

244. From each column subtract the preceding one. In the resulting determinant, subtract from each column the preceding one, keeping the first two fixed. Again, subtract from each column the preceding one, keeping the first three columns fixed, and so on. After m such operations we get a determinant in which all elements of the last column are 1. The evaluation of this determinant presents no special difficulties.

245. From each row subtract the preceding one and show that $\Delta_{n+1}=(x-1)\Delta_n$.

246. From each row subtract the preceding one and show that $\Delta_{n+1}=(n-1)!\,(x-1)\Delta_n$.

247. From each row subtract the preceding one; from each column subtract the preceding one. Prove that $\Delta_n=\alpha\Delta_{n-1}$.

248. Represent the last element of the last row as $z+(x-z)$. Represent the determinant as a sum of two determinants. Use the fact that the determinant is symmetric in y and z.

249. See the suggestion of Problem 248.

252. Subtract from each row the first row multiplied by $\frac{\beta}{a}$. In the resulting determinant, take $\frac{ab-\lambda\beta}{a(\alpha-\beta)}$ out of the first column and subtract from the first column all the other columns.

253. Add all columns to the first and from each row subtract the preceding one. See Problem 199.

254. Use the suggestion of Problem 253.

256. Regard the determinant as a polynomial in a of degree four. Show that the desired polynomial is divisible by the following linear polynomials in a:

$$a+b+c+d,\ a+b-c-d,\ a-b+c-d,\ a-b-c+d.$$

258. Adding all columns to the first, separate out the factor $x+a_1+\ldots+a_n$ Then setting $x=a_1, a_2, \ldots, a_n$, convince yourself that the determinant is divisible by $x-a_1, x-a_2, \ldots, x-a_n$.

259. The Vandermonde determinant.

264. Expand by elements of the first column.

265. From the second row subtract the first. In the resulting determinant subtract the second row from the third, etc.

269. Take $\frac{1}{2!}$ out of the third row, $\frac{1}{3!}$ out of the fourth, etc.

270. Make use of the result of Problem 269.

271. Take 2 out of the second column, 3 out of the third, etc. When computing $\prod\limits_{n\geqslant i>k\geqslant 1}(i^2-k^2)$ it is useful to represent

$$\prod(i^2-k^2)=\prod(i-k)\cdot\prod(i+k).$$

272. Take $\frac{x_1}{x_1-1}$ out of the first column, $\frac{x_2}{x_2-1}$ out of the second, etc.

273. Take a_1^n out of the first row, a_2^n out of the second, and so on.

275. To the first column add the second multiplied by $C^1{}_n$, the third multiplied by C^2_{2n}, etc.

276. Take advantage of the result of Problem 51.

277. Take advantage of Problem 53.

278. Adjoin the row $1, x_1, x_2, \ldots, x_n$ and the column $1, 0, 0, \ldots, 0$.

279. Consider the determinant

$$D=\begin{vmatrix} 1 & 1 & \dots & 1 & 1 \\ x_1 & x_2 & \dots & x_n & z \\ x_1^2 & x_2^2 & \dots & x_n^2 & z^2 \\ \dots & \dots & \dots & \dots & \dots \\ x_1^n & x_2^n & \dots & x_n^n & z^n \end{vmatrix}.$$

Compare the expansion of D by elements of the last column with the expression $D=\prod\limits_{n\geqslant i>k\geqslant 1}(x_i-x_k)\cdot\prod\limits_{i=1}^{n}(z-x_i)$.

280. Use the suggestion of Problem 279.

282. Adjoin the first row 1, 0, ..., 0 and thefi rst column 1, 1, 1, ..., 1. Subtract the first column from all the succeeding columns.

285. Expand by elements of the last row.

286. First, from each column (beginning with the last) subtract the preceding column multiplied by x. Then, after reducing the order and taking out obvious factors, transform the first rows (dependent on x) using the relation

$$(m+1)^s-m^s=sm^{s-1}+\frac{s(s-1)}{2}m^{s-2}+\dots+1.$$

287. From each column, beginning with the last, subtract the preceding column multiplied by x.

288. (m) Add to the first column the sixth and the eleventh, to the second column, the seventh and the twelfth, ..., to the fifth column, the tenth and the fifteenth. Add to the sixth column the eleventh, to the seventh column, the twelfth, ..., to the tenth column, the fifteenth.

From the fifteenth row subtract the tenth, from the fourteenth row subtract the ninth, ..., from the sixth row subtract the first.

293. Consider

$$\begin{vmatrix} 1 & C_n^1 a_0 & C_n^2 a_0^2 & \dots & a_0^n \\ 1 & C_n^1 a_1 & C_n^2 a_1 & \dots & a_1^n \\ \dots & \dots & \dots & \dots & \dots \\ 1 & C_n^1 a_n & C_n^2 a_n^2 & \dots & a_n^n \end{vmatrix}\cdot\begin{vmatrix} b_0^n & b_1^n & \dots & b_n^n \\ b_0^{n-1} & b_1^{n-1} & \dots & b_n^{n-1} \\ \dots & \dots & \dots & \dots \\ 1 & 1 & \dots & 1 \end{vmatrix}.$$

294. Consider

$$\begin{vmatrix} \sin\alpha_1 & \cos\alpha_1 & 0 & \dots & 0 \\ \sin\alpha_2 & \cos\alpha_2 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \sin\alpha_n & \cos\alpha_n & 0 & \dots & 0 \end{vmatrix}\cdot\begin{vmatrix} \cos\alpha_1 & \cos\alpha_2 & \dots & \cos\alpha_n \\ \sin\alpha_1 & \sin\alpha_2 & \dots & \sin\alpha_n \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{vmatrix}.$$

295. Consider

$$\begin{vmatrix} 1 & 1 & \dots & 1 & 1 \\ x_1 & x_2 & \dots & x_n & x \\ \dots & \dots & \dots & \dots & \dots \\ x_1^n & x_2^n & \dots & x_n^n & x^n \end{vmatrix} \cdot \begin{vmatrix} 1 & x_1 & \dots & x_1^{n-1} & 0 \\ 1 & x_2 & \dots & x_2^{n-1} & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & x_n & \dots & x_n^{n-1} & 0 \\ 0 & 0 & \dots & 0 & 1 \end{vmatrix}.$$

296. Raise to the second power.

297. Subtract from the third column the first, from the fourth the second. Then multiply by

$$\begin{vmatrix} \cos\varphi & -\sin\varphi & 0 & 0 \\ \sin\varphi & \cos\varphi & 0 & 0 \\ 0 & 0 & \cos 2\varphi & -\sin 2\varphi \\ 0 & 0 & \sin 2\varphi & \cos 2\varphi \end{vmatrix}.$$

298. Subtract from the second column n times the first, from the fourth, n times the second. Interchange the second and third columns. Multiply by

$$\begin{vmatrix} \cos n\varphi & -\sin n\varphi & 0 & 0 \\ \sin n\varphi & \cos n\varphi & 0 & 0 \\ 0 & 0 & \cos(n+1)\varphi & -\sin(n+1)\varphi \\ 0 & 0 & \sin(n+1)\varphi & \cos(n+1)\varphi \end{vmatrix}.$$

299. Square it. Transform as a Vandermonde determinant and transform each difference to the sine of some angle. This will yield the sign.

300. Study the product

$$\begin{vmatrix} a_0 & a_1 & a_2 & \dots & a_{n-1} \\ a_{n-1} & a_0 & a_1 & \dots & a_{n-2} \\ \dots & \dots & \dots & \dots & \dots \\ a_1 & a_2 & a_3 & \dots & a_0 \end{vmatrix} \cdot \begin{vmatrix} 1 & 1 & \dots & 1 \\ 1 & \varepsilon_1 & \dots & \varepsilon_{n-1} \\ \dots & \dots & \dots & \dots \\ 1 & \varepsilon_1^{n-1} & \dots & \varepsilon_{n-1}^{n-1} \end{vmatrix}$$

where $\varepsilon_k = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}$.

308. Consider $\varepsilon_1 = \cos\frac{\pi}{n} + i\sin\frac{\pi}{n}$. Then

$$\Delta = \prod_{k=0}^{n-1} \left(\sum_{j=1}^{n} \frac{\varepsilon_1^j + \varepsilon_1^{-j}}{2} \varepsilon_1^{2k(j-1)} \right).$$

311. Use Problem 92.

$$\textbf{314.}\ \prod_{k=0}^{2n-1}(a_0+a_1\varepsilon_k+a_2\varepsilon_k^2+\ldots+a_{2n-1}\varepsilon_k^{2n-1})$$
$$=\prod_{r=0}^{n-1}[(a_0+a_n)+(a_1+a_{n+1})\alpha_r+\ldots+(a_{n-1}+a_{2n-1})\alpha_r^{n-1}]$$
$$\times\prod_{s=0}^{n-1}[(a_0-a_n)+(a_1-a_{n+1})\beta_s+\ldots+(a_{n-1}-a_{2n-1})\beta_s^{n-1}]$$

where $\varepsilon_k=\cos\frac{k\pi}{n}+i\sin\frac{k\pi}{n}$; $\alpha_r=\cos\frac{2r\pi}{n}+i\sin\frac{2r\pi}{n}$;

$$\beta_s=\cos\frac{(2s+1)\pi}{n}+i\sin\frac{(2s+1)\pi}{n}.$$

323. From each row subtract the first, from each column subtract the first.

325. Use Problem 217.

327. Represent in the form of a sum of determinants or set $x=0$ in the determinant and its derivatives.

328. (1) From the $(2n-1)$ th row subtract the $(2n-2)$th, from the $(2n-2)$th row subtract the $(2n-3)$th, ..., and from the $(n+1)$th row subtract the nth, from the nth row subtract the sum of all the preceding ones.

(2) Add to the $(n+i)$th row the ith, $i=1, 2, \ldots, n-1$.

329. Add to every row all the subsequent ones, and subtract from every column the preceding column. Prove that

$$\Delta_{n+1}(x)=(x-n)\Delta_n(x-1).$$

CHAPTER 4

MATRICES

466. Use the result of Problem 465 (e).

473. Consider the sum of the diagonal elements.

491. Take advantage of the results of Problems 489, 490.

492. Use the result of Problem 490.

494, 495. Use the results of Problems 492, 493.

496. Argue by induction with respect to the number of columns of the matrix B, first having proved that if the adjoining of one column does not change the rank of B, then it does not change the rank of the matrix (A, B) either.

A proof other than by induction can be carried out by using the Laplace theorem.

497. Take advantage of the results of Problems 496, 492.

498. From the matrix $(E-A, E+A)$ select a nonsingular square matrix P and consider the product $(E-A)P$ and $(E+A)P$.

500. Take advantage of the result of Problem 489.

501. Prove the uniqueness of the representation in Problem 500 and thus reduce the problem to counting the number of triangular matrices R with a given determinant k. Denoting the desired number by $F_n(k)$, prove that if $k=a\cdot b$ for relatively prime a, b, then $F_n(k)=F_n(a)\,F_n(b)$. Finally, construct, inductively, a formula for $F_n(p^m)$, where p is prime.

505. Take advantage of the results of Problems 495, 498. Find the matrix P with the smallest possible determinant so that $P^{-1}AP$ is diagonal, and then use the result of Problem 500.

517. Use the Laplace theorem and the Bunyakovsky inequality.

518. Establish the equation $|\bar{A}A|=|\bar{B}B|\cdot|\bar{C}C|$ on the assumption that the sum of the products of the elements of any column of matrix B into corresponding elements of any column of matrix C is equal to zero. Then complete (in appropriate fashion) the matrix (B, C) to a square matrix and take advantage of the result of Problem 517.

523. On the left of the determinant, adjoin a column, all elements of which are equal to $\frac{M}{2}$; adjoin at the top a row, all elements of which (except the corner) are 0; then subtract the first column from all other columns.

527. Take advantage of the results of Problems 522, 526.

528. Establish a connection between an adjoint matrix and an inverse matrix.

529. With respect to the minor formed from elements of the first m rows and the first m columns of the adjoint matrix, establish the result by considering the product of the matrices

$$\begin{pmatrix} A_{11} & \dots & A_{m+1,1} & \dots & A_{n1} \\ A_{12} & \dots & A_{m+1,2} & \dots & A_{n2} \\ \dots & \dots & \dots & \dots & \dots \\ A_{1m} & \dots & A_{m+1,m} & \dots & A_{nm} \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} \cdot \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

where A_{ik} are the cofactors of the elements a_{ik}.

Do the same for the general case.

535. Represent $A\times B$ as $(A\times E_m)\cdot(E_n\times B)$.

537. First analyze the case when A_{11} is a nonsingular matrix and then argue by induction. Reduce the general case to this case, adding λE to the matrix.

CHAPTER 5

POLYNOMIALS AND RATIONAL FUNCTIONS OF ONE VARIABLE

547. (a) Expand $f(x)$ in powers of $x-3$, then substitute $x+3$ for x.

553. Differentiate directly and substitute $x=1$, then isolate the maximum power of x and continue the differentiation.

555. Consider the polynomials

$$f_1(x)=nf(x)-xf'(x),\quad f_2(x)=nf_1(x)-xf_1'(x)$$

and so on.

561. Prove by the method of mathematical induction.

562. The nonzero root of multiplicity $k-1$ of the polynomial $f(x)$ is a root of multiplicity $k-2$ of the polynomial $xf'(x)$, a root of multiplicity $k-3$ of the polynomial $x[xf'(x)]'$, etc.

Conversely, the general nonzero root of the polynomials $f(x)$, $xf'(x)$, $x[xf'(x)]'$, ... (a total of $k-1$ polynomials) is a root of $f(x)$ of multiplicity not lower than $k-1$.

563. Differentiate the equation showing that the polynomial is divisible by its derivative.

567. Consider the function $\dfrac{f_1(x)}{f_2(x)}$ or $\dfrac{f_2(x)}{f_1(x)}$.

568. Relate the problem to a consideration of the roots

$$\varphi(x)=f(x)f'(x_0)-f'(x)f(x_0)$$

where x_0 is a root of $[f'(x)]^2-f(x)f''(x)$.

569. Use the solution of Problem 568 and expand $f(x)$ in powers of $x-x_0$.

576. Prove it like d'Alembert's lemma.

580, 581. Represent the function in the same form as in proving the d'Alembert lemma:

$$f(z)=f(a)+\frac{f^k(a)}{k!}(z-a)^k[1+\varphi(z)], \quad \varphi(a)=0.$$

583. Find the roots of the polynomials and take into account the leading coefficients [in Problems (a) and (b)]. It is advisable, in Problem (c), to set $x=\tan^2\theta$ when seeking the roots.

589. Find the common roots.

608. First prove that $f(x)$ does not have any real roots of odd multiplicity.

623. Use the result of Problem 622.

626. Use the fact that the equation should not change when $-x$ is substituted for x and $\dfrac{1}{x}$ for x.

627. The equation should not change when $\dfrac{1}{x}$ is substituted for x and $1-x$ for x.

637. Divide by $(1-x)^n$ and differentiate $m-1$ times, assuming $x=0$ after each differentiation. Take advantage of the fact that the degree of $N(x)$ is less than m and the degree of $M(x)$ is less than n.

642. Use the Lagrange formula. Perform the division in each term of the result and collect like terms using the result of Problem 100.

644. Express $f(x_0)$ in terms of $f(x_1), f(x_2), \ldots, f(x_n)$, using the Lagrange interpolation formula and compare the result with the hypothesis of the problem, taking into account the independence of $f(x_1), f(x_2), \ldots, f(x_n)$. Then study $\varphi(x)=(x-x_1)(x-x_2)\ldots(x-x_n)$, expanding it in powers of $x-x_0$.

645. Represent the polynomial x^s in terms of its values by means of the Lagrange interpolation formula.

648, 649. Construct an interpolation polynomial by Newton's method.

650. Find the values of the desired polynomial for $x=0, 1, 2, 3, \ldots, 2n$.

651. The problem can be solved by using Newton's method. A shorter way is to consider the polynomial $F(x)=xf(x)-1$, where $f(x)$ is the desired polynomial.

652. Consider the polynomial $(x-a)f(x)-1$.

653. Construct the polynomial by Newton's method and, for convenience of computation, introduce a factorial into the denominator of each term.

654. Consider the polynomial $f(x^2)$, where $f(x)$ is the desired polynomial.

655. The easiest way is by the Lagrange formula

$$\frac{f(x)}{\varphi(x)}=\sum_{k=1}^{n}\frac{f(x_k)}{(x-x_k)\,\varphi'(x_k)}$$

($x_1, x_2, \ldots, x_n$ are roots of the denominator).

656. First expand by the Lagrange formula, then combine complex conjugate terms.

657. (e) Use Problem 631. (f) Set $\frac{a+x}{2a}=y$. (d), (h) Seek expansions by the method of undetermined coefficients. Find part by the substitution $x=x_1, x_2, \ldots, x_n$ after multiplying by the common denominator. Then differentiate and again set $x=x_1, x_2, \ldots, x_n$.

660. Use Problem 659. In Problem (b) decompose $\frac{1}{x^2-3x+2}$ into partial fractions.

665, 666. Take advantage of Problem 663.

667. In Problem (c) expand the polynomial in powers of $x-1$.

668. Expand in powers of $x-1$ (or put $x=y+1$).

669. Set $x=y+1$ and use mathematical induction to prove that all coefficients of the dividend and divisor (except the leading coefficients) are divisible by p.

670, 671. The proof is like that of the Eisenstein theorem.

679, 680. Assuming reducibility of $f(x)$, set $x=a_1, a_2, \ldots, a_n$ and draw a conclusion concerning the values of the divisors.

681. Count the number of equal values of the presumed divisors.

682. Use the fact that $f(x)$ does not have real roots.

683. Prove that a polynomial having more than three integral roots cannot have for one of its values a prime in the case of an integral value of the independent variable; apply this to the polynomial $f(x)-1$.

684, 685. Use the result of Problem 683.

702. Construct a Sturm sequence and consider separately the cases of even and odd n.

707-712. Derive recurrence relations between the polynomials of consecutive degrees and their derivatives and use them to construct a Sturm sequence. In Problem 708, construct a Sturm sequence solely for positive values of x and use other reasoning to assure yourself that there are no negative roots. In Problem 709, construct a Sturm sequence for negative x.

713. Use the fact that $F'(x)=2f(x)f'''(x)$ and that $f'''(x)$ is a constant.

717. Factor $g(x)$ and use the result of Problem 716 several times.

718. Apply the result of Problem 717 to the polynomial x^m.

719. Use the fact that if all the roots of the polynomial $a_0x^n+a_1x^{n-1}+\ldots+a_{n-1}x+a_n$ are real, then all the roots of the polynomial $a_nx^n+a_{n-1}x^{n-1}+\ldots+a_0$ are real.

721. Multiply by $x-1$.

727. Prove by contradiction by taking advantage of Rolle's theorem and the result of Problem 581.

728. Construct the graph of $\psi(x)=\dfrac{f(x)}{f'(x)}$ and prove rigorously that every root of $[f'(x)]^2-f(x)f''(x)$ yields an extreme point for $\psi(x)$ and conversely. Prove that $\psi(x)$ has no extreme points in the intervals, between the roots of $f'(x)$, that contain a root of $f(x)$, and has exactly one extreme point in the intervals which do not contain roots of $f(x)$.

729. Use the result of Problems 727 and 726.

730. Study the behaviour of the function

$$\psi(x)=\frac{f(x)}{f'(x)}+\frac{x+\lambda}{\gamma}.$$

731. It is solved on the basis of the preceding problem for $\lambda=0$.

732. Prove by means of induction with respect to the degree of $f(x)$, setting $f(x)=(x+\lambda)f_1(x)$, where $f_1(x)$ is a polynomial of degree $n-1$.

733. The proof is obtained by a double application of the result of Problem 732.

734. If all the roots of $f(x)$ are positive, then the proof is effected by elementary means, namely by induction with respect to the degree of $f(x)$. Include in the induction hypotheses that the roots $x_1, x_2, \ldots, x_{n-1}$ of the polynomial $b_0+b_1wx+\ldots+b_{n-1}w^{(n-1)^2}x^{n-1}$ satisfy the condition

$$0<x_1<x_2<\ldots<x_{n-1} \quad \text{and} \quad x_i>x_{i-1}w^{-2}.$$

To prove the theorem in the general case, it is necessary to represent w^{x^2} as the limit of a polynomial in x with roots not contained in the interval $(0, n)$ and to take advantage of the result of Problem 731.

735. Consider $\left|\dfrac{\varphi(x)+i\psi(x)}{\varphi(x)-i\psi(x)}\right|$, where

$$\varphi(x)=a_0\cos\varphi+\ldots+a_n\cos(\varphi+n\theta)x^n,$$
$$\psi(x)=b_0\sin\varphi+\ldots+b_n\sin(\varphi+n\theta)x^n.$$

736. Consider the modulus of $\dfrac{\varphi(x)+i\psi(x)}{\varphi(x)-i\psi(x)}$, where

$$\varphi(x)=a_0+a_1x+\ldots+a_nx^n,$$
$$\psi(x)=b_0+b_1x+\ldots+b_nx^n.$$

Having proved the real nature of the roots, multiply $\varphi(x)+i\psi(x)$ by $\alpha-\beta i$ and consider the real part. Use the result of Problem 727.

737. Decompose $\dfrac{\psi(x)}{\varphi(x)}$ into partial fractions, investigate the signs of the coefficients in this decomposition and study the imaginary part

$$\frac{-i[\varphi(x)+i\psi(x)]}{\varphi(x)}=\frac{\psi(x)}{\varphi(x)}-i.$$

738. Investigate the imaginary part of $\dfrac{f'(x)}{f(x)}$ by decomposing the fraction into partial fractions.

739. Change the variable so that the given half-plane is converted into the half-plane $\operatorname{Im}(x)>0$.

740. Relate to Problem 739.

741. Decompose $\frac{f'(x)}{f(x)}$ into partial fractions and estimate the imaginary part.

743. Set $x=yi$ and take advantage of the results of Problems 736 and 737.

744, 745. Take advantage of the result of Problem 743.

746. Set $x=\frac{1+y}{1-y}$ and use the result of Problem 744.

747. Multiply the polynomial by $1-x$ and, setting $|x|=\rho>1$, estimate the modulus of $(1-x)f(x)$.

CHAPTER 6

SYMMETRIC FUNCTIONS

772. The sides of a triangle similar to the given one and inscribed in a circle of radius $\frac{1}{2}$ are equal to the sines of the angles of the given triangle.

800. First compute the sum

$$\sum_{i=1}^{n} (x+x_i)^k$$

and then substitute $x=x_j$ and sum from 1 to n with respect to j. Finally, delete extraneous terms and divide by 2.

801. The solution is like that of Problem 800.

805. Every primitive nth root of unity raised to the mth power yields a primitive root of degree $\frac{n}{d}$, where d is the greatest common divisor of m and n. As a result of this operation performed with respect to all primitive nth roots of unity, all primitive roots of degree $\frac{n}{d}$ are obtained the same number of times.

806. Use the results of Problems 805, 117, and 119.

807. It is necessary to find an equation whose roots are $x_1, x_2, \ldots, x_n$. To do this, use Newton's formulas or a representation of the coefficients in terms of power sums in the form of a determinant (Problem 803).

808. The problem is readily solved by means of Newton's formulas or by means of representing power sums in terms of the elementary symmetric functions in the form of determinants (Problem 802). However, it is still easier to multiply the equation by $(x-a)\cdot(x-b)$ and compute the power sums for the new equation.

809. The simplest way is to multiply the equation by $(x-a)\cdot(x-b)$.

818. Consider the roots of the polynomial $f(x)$ as independent variables. Multiply the determinant of the coefficients of the remainders by the Vandermonde determinant.

819. First prove that all polynomials ψ_k have degree $n-1$. Then multiply the determinant of the coefficients of ψ_k by the Vandermonde determinant.

820. The solution is like that of Problem 819.

827. Use the fact that the mth degrees of the primitive nth roots of 1 run through all primitive roots of unity of degree $\frac{n}{d}$, where d is the greatest common divisor of m and n.

828. Use the result of Problem 827 and the fact that $R(X_m, X_n)$ is a divisor of $R(X_m, x^n-1)$ and $R(X_n, x^m-1)$.

834, 835. Compute $R(f', f)$.

839. Multiply by $x-1$.

840. Multiply by $x-1$ and use the result of Problem 835.

843. Compute $R(X_n, X_n')$. In computing the values of X_n' for the roots of X_n, represent X_n in the form

$$(x^n-1)\,\Pi\,(x^d-1)^{\mu\left(\frac{n}{d}\right)}$$

considering that d runs through the proper divisors of n.

844. Take advantage of the relation $E_n'=E_n-x^n$.

845. Take advantage of the relation

$$(nx-x-a)\,F_n-x\,(x+1)\,F_n'+\frac{(a-1)\ \dots\ (a-n)}{n!}=0.$$

846. Use the relations

$$P_n=xP_{n-1}-(n-1)\,P_{n-2},\ P_n'=nP_{n-1}.$$

847. Use the relations

$$xP_n'=nP_n+n^2P_{n-1},\ P_n=(x-2n+1)\,P_{n-1}-(n-1)^2\,P_{n-2}.$$

848. Use the relations

$$(4-x^2)\,P_n'+nxP_n=2n\,P_{n-1},\ P_n-xP_{n-1}+P_{n-2}=0.$$

849. Use the relations

$$P_n-2xP_{n-1}+(x^2+1)\,P_{n-2}=0,\ P_n'=(n+1)\,P_{n-1}.$$

850. Use the relations

$$P_n-(2n-1)\,xP_{n-1}+(n-1)^2\,(x^2+1)\,P_{n-2}=0,\ P_n'=n^2P_{n-1}.$$

851. Use the relations

$$P_n-(2nx+1)\,P_{n-1}+n\,(n-1)\,x^2P_{n-2}=0,$$
$$P_n'=(n+1)\,nP_{n-1}.$$

852. Solve the problem by Lagrange's method of multipliers. Write the result of equating the derivatives to zero in the form of a differential equation with respect to the polynomial that yields a maximum, and solve the equation by the method of undetermined coefficients.

867. First demonstrate that there is only a finite number of equations with the given properties for a given n. Then show that the properties are not destroyed under the transformation $y=x^m$.

CHAPTER 7

LINEAR ALGEBRA

884. Use the results of Problems 51, 52.

916. The smallest angle is to be sought among the angles formed by vectors of the second plane with their orthogonal projections on the first plane.

917. Specify the cube in a system of coordinates with origin at the centre and with axes parallel to the edges. Then take four mutually orthogonal diagonals for the axes.

918. Use the result of Problem 907.

920. Prove by induction.

921. Use the fact that $V[A_1, \ldots, A_m, B_1, \ldots, B_k] = V[A_1, \ldots, A_m] \cdot V[B_1, \ldots, B_k]$ if $A_i \perp B_j$ and use the result of the preceding problem.

933. First find the eigenvalues of the square of the matrix. Then, to determine the signs in taking the square root, use the fact that the sum of the eigenvalues is equal to the sum of the elements of the principal diagonal and that the product of the eigenvalues is equal to the determinant. Apply the results of Problems 126 and 299.

934. Apply the result of Problem 933.

936. Use the results of Problems 537 and 930.

943. (1) Use the fact that the determinant of a triangular transformation is equal to unity.

(2) Set

$$x_{k+1} = x_{k+2} = \ldots = x_n = 0.$$

945. For the new independent variable take the linear form whose square is added to the quadratic form.

946. Isolate one square from the form f and use the result of Problem 945.

948. Consider the quadratic form in the unknowns $u_1, u_2, \ldots, u_n$:

$$f = \sum_{k=1}^{n} (u_1 + u_2 x_k + \ldots + u_n x_k^{n-1})^2$$

where $x_1, x_2, \ldots, x_n$ are roots of the given equation.

950. Decompose f and φ into a sum of squares and use the distributivity of the operation (f, φ).

965. In proving the converse, make use of the factorization $X = AX + (E-A)X$.

966. Write the projection matrix in the basis obtained by combining the orthonormal bases P and Q.

967. Be sure that $AX \cdot X = 0$ for any real vector X. Decompose the eigenvalue and eigenvector into a real part and an imaginary part.

968. Multiply the matrix A on the right by P, on the left by P^{-1}, where P

is an orthogonal matrix, the first two columns of which are composed of normalized real and imaginary parts of the eigenvector.

970, 971. Use the fact that for the orthogonal matrix A, $AX \cdot AY = X \cdot Y$ for any real vectors X and Y.

972. The proof is based on the results of Problems 970, 971 and, like Problem 968, on the results of Problem 967.

975, 976. Go to the Jordan canonical form.

978. Connect it with the solution of Problem 977.

980. For necessity, see Problem 473.

For the sufficiency proof, consider first the case when all diagonal elements of the matrix C are zero. Then use the fact that if $C = XY - YX$, then

$$S^{-1}CS = (S^{-1}XS)(S^{-1}YS) - (S^{-1}YS)(S^{-1}XS).$$

PART III. ANSWERS AND SOLUTIONS

CHAPTER 1

COMPLEX NUMBERS

1. $x=-\frac{4}{11}, \quad y=\frac{5}{11}$.

2. $x=-2, \quad y=\frac{3}{2}, \quad z=2, \quad t=-\frac{1}{2}$.

3. 1 if $n=4k$; i if $n=4k+1$; -1 if $n=4k+2$; $-i$ if $n=4k+3$; k an integer.
5. (a) $117+44i$, (b) -556, (c) $-76i$.
6. If and only if:
(1) none of the factors is zero;
(2) the factors are of the form $(a+bi)$ and $\lambda(b+ai)$, where λ is a real number.

7. (a) $\cos 2\alpha+i\sin 2\alpha$, (b) $\frac{a^2-b^2}{a^2+b^2}+i\frac{2ab}{a^2+b^2}$, (c) $\frac{44-5i}{318}$,

(d) $\frac{-1-32i}{25}$, (e) 2.

8. $2i^{n-1}$.

9. (a) $x=1+i$, $y=i$; (b) $x=2+i$, $y=2-i$; (c) $x=3-11i$, $y=-3-9i$, $z=1-7i$.

10. (a) $-\frac{1}{2}-i\frac{\sqrt{3}}{2}$, (b) 1.

11. (a) $a^2+b^2+c^2-(ab+bc+ac)$; (b) a^3+b^3;
(c) $2(a^3+b^3+c^3)-3(a^2b+a^2c+b^2a+b^2c+c^2a+c^2b)+12abc$;
(d) a^2-ab+b^2.

12. (a) $0,\ 1,\ -\frac{1}{2}+\frac{i\sqrt{3}}{2},\ -\frac{1}{2}-\frac{i\sqrt{3}}{2}$; (b) $0,\ 1,\ i,\ -1,\ -i$.

15. (a) $\pm(1+i)$; (b) $\pm(2-2i)$; (c) $\pm(2-i)$; (d) $\pm(1+4i)$;
(e) $\pm(1-2i)$; (f) $\pm(5+6i)$; (g) $\pm(1+3i)$; (h) $\pm(1-3i)$;

(i) $\pm(3-i)$; (j) $\pm(3+i)$; (k) $\pm\left(\sqrt{\frac{\sqrt{13}+2}{2}}-i\sqrt{\frac{\sqrt{13}-2}{2}}\right)$;

(l) $\pm\sqrt{8+2\sqrt{17}}\pm i\sqrt{-8+2\sqrt{17}}$; (m) $\pm\left(\sqrt{\frac{3}{2}}-i\sqrt{\frac{1}{2}}\right)$;

(n) $\frac{\sqrt{2}(\pm 1\pm i)}{2}$; (o) $i^{\alpha}\left(\frac{1+\sqrt{3}}{2}+\frac{1-\sqrt{3}}{2}i\right)$, $\alpha=0,\ 1,\ 2,\ 3.$

16. $\pm(\beta-\alpha i)$.

17. (a) $x_1=3-i$, $x_2=-1+2i$; (b) $x_1=2+i$, $x_2=1-3i$;

(c) $x_1=1-i$, $x_2=\frac{4-2i}{5}$.

18. (a) $1\pm 2i$, $-4\pm 2i$, $(x^2-2x+5)(x^2+8x+20)$;

(b) $2\pm i\sqrt{2}$, $-2\pm 2i\sqrt{2}$, $(x^2-4x+6)(x^2+4x+12)$.

19. (a) $x=\pm\frac{\sqrt{7}}{2}\pm\frac{i}{2}$; (b) $\pm 4\pm i$.

20. $\pm\sqrt{\frac{\sqrt{q}}{2}-\frac{p}{4}}\pm i\sqrt{\frac{\sqrt{q}}{2}+\frac{p}{4}}$.

22. (a) $\cos 0+i\sin 0$; (b) $\cos\pi+i\sin\pi$; (c) $\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}$;

(d) $\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}$; (e) $\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$;

(f) $\sqrt{2}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)$; (g) $\sqrt{2}\left(\cos\frac{5\pi}{4}+i\sin\frac{5\pi}{4}\right)$;

(h) $\sqrt{2}\left(\cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4}\right)$; (i) $2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$;

(j) $2\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)$; (k) $2\left(\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}\right)$;

(l) $2\left(\cos\frac{5\pi}{3}+i\sin\frac{5\pi}{3}\right)$; (m) $2\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)$;

(n) $3(\cos\pi+i\sin\pi)$; (o) $2\left(\cos\frac{11\pi}{6}+i\sin\frac{11\pi}{6}\right)$;

(p) $(\sqrt{2}+\sqrt{6})\left(\cos\frac{\pi}{12}+i\sin\frac{\pi}{12}\right)$.

Remark. Given here is one of the possible values of the argument.

23. (a) $\sqrt{10}\,(\cos 18°26'+i\sin 18°26')$;

(b) $\sqrt{17}\,(\cos 345°57'48''+i\sin 345°57'48'')$;

(c) $\sqrt{5}$ $(\cos 153°26'6''+i \sin 153°26'6'')$;

(d) $\sqrt{5}$ $(\cos 243°26'6''+i \sin 243°26'6'')$.

24. (a) A circle of radius 1 with centre at the origin.

(b) A ray issuing from the origin at an angle of $\frac{\pi}{6}$ to the positive direction of the axis of reals.

25. (a) The interior of a circle of radius 2 with centre at the coordinate origin.

(b) The interior and contour of a circle of radius 1 with centre at the point (0, 1).

(c) The interior of a circle of radius 1 with centre at the point (1, 1).

26. (a) $x=\frac{3}{2}-2i$, (b) $x=\frac{3}{4}+i$.

27. The identity expresses a familiar theorem of geometry: the sum of the squares of the diagonals of a parallelogram is equal to the sum of the squares of its sides.

29. If the difference of the arguments of these numbers is equal to $\pi+2k\pi$, where k is an integer.

30. If the difference of the arguments of these numbers is equal to $2k\pi$, where k is an integer.

34. $\cos(\varphi+\psi)+i \sin(\varphi+\psi)$.

35. $\frac{\sqrt{2}}{2}\left[\cos\left(2\varphi-\frac{\pi}{12}\right)+i \sin\left(2\varphi-\frac{\pi}{12}\right)\right]$.

36. (a) $2^{12}(1+i)$, (b) $2^{9}(1-i\sqrt{3})$, (c) $(2-\sqrt{3})^{12}$, (d) -64.

38. $\cos\frac{n\pi}{3}+i \sin\frac{n\pi}{3}$.

39. $2\cos\frac{2n\pi}{3}$.

40. *Solution.* $1+\cos\alpha+i\sin\alpha$

$$=2\cos^2\frac{\alpha}{2}+2i\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}=2\cos\frac{\alpha}{2}\left(\cos\frac{\alpha}{2}+i\sin\frac{\alpha}{2}\right);$$

$$(1+\cos\alpha+i\sin\alpha)^n=2^n\cos^n\frac{\alpha}{2}\left(\cos\frac{n\alpha}{2}+i\sin\frac{n\alpha}{2}\right).$$

43. (a) $-i$, $\frac{\sqrt{3}+i}{2}$, $\frac{-\sqrt{3}+i}{2}$;

(b) $-1+i$, $\frac{1+\sqrt{3}}{2}+\frac{\sqrt{3}-1}{2}i$, $\frac{1-\sqrt{3}}{2}-\frac{1+\sqrt{3}}{2}i$;

(c) $1+i$, $1-i$, $-1+i$, $-1-i$;

(d) 1, -1, $-\frac{1}{2}+i\frac{\sqrt{3}}{2}$, $-\frac{1}{2}-i\frac{\sqrt{3}}{2}$, $\frac{1}{2}+i\frac{\sqrt{3}}{2}$, $\frac{1}{2}-i\frac{\sqrt{3}}{2}$;

(e) $i\sqrt{3}$, $-i\sqrt{3}$, $\frac{3+i\sqrt{3}}{2}$, $\frac{3-i\sqrt{3}}{2}$, $-\frac{3+i\sqrt{3}}{2}$, $\frac{-3+i\sqrt{3}}{2}$.

44. (a) $\sqrt[6]{5}\ (\cos 8°5'18''+i\sin 8°5'18'')\ \varepsilon_k$,
where $\varepsilon_k=\cos 120°k+i\sin 120°k$, $k=0, 1, 2$;

(b) $\sqrt[6]{10}\ (\cos 113°51'20''+i\sin 113°51'20'')\ \varepsilon_k$,
where $\varepsilon_k=\cos 120°k+i\sin 120°k$, $k=0, 1, 2$;

(c) $\sqrt[10]{13}\ (\cos 11°15'29''+i\sin 11°15'29'')\ \varepsilon_k$,
where $\varepsilon_k=\cos 72°k+i\sin 72°k$, $k=0, 1, 2, 3, 4$.

45. (a) $\dfrac{1}{\sqrt[12]{2}}\left(\cos\dfrac{24k+19}{72}\pi+i\sin\dfrac{24k+19}{72}\pi\right)$,

where $k=0, 1, 2, 3, 4, 5$;

(b) $\dfrac{1}{\sqrt[16]{2}}\left(\cos\dfrac{24k+5}{96}\pi+i\sin\dfrac{24k+5}{96}\pi\right)$,

where $k=0, 1, 2, 3, 4, 5, 6, 7$;

(c) $\dfrac{1}{\sqrt[12]{2}}\left(\cos\dfrac{24k+17}{72}\pi+i\sin\dfrac{24k+17}{72}\pi\right)$,

where $k=0, 1, 2, 3, 4, 5$.

46. $\beta\left(\cos\dfrac{2k\pi}{n}+i\sin\dfrac{2k\pi}{n}\right)$, where $k=0, 1, 2, \ldots, n-1$.

47. (a) *Solution.* Consider $(\cos x+i\sin x)^5$. By De Moivre's formula,

$$(\cos x+i\sin x)^5=\cos 5x+i\sin 5x.$$

On the other hand,

$(\cos x+i\sin x)^5=\cos^5 x+5i\cos^4 x\sin x-10\cos^3 x\sin^2 x-10i\cos^2 x$
$\sin^3 x+5\cos x\sin^4 x+i\sin^5 x=(\cos^5 x-10\cos^3 x\sin^2 x+5\cos x\sin^4 x)$
$+i(5\cos^4 x\sin x-10\cos^2 x\sin^3 x+\sin^5 x)$.

Comparing the results, we have

$\cos 5x=\cos^5 x-10\cos^3 x\sin^2 x+5\cos x\sin^4 x$;

(b) $\cos^8 x-28\cos^6 x\sin^2 x+70\cos^4 x\sin^4 x-28\cos^2 x\sin^6 x+\sin^8 x$;

(c) $6\cos^5 x\sin x-20\cos^3 x\sin^3 x+6\cos x\sin^5 x$;

(d) $7\cos^6 x\sin x-35\cos^4 x\sin^3 x+21\cos^2 x\sin^5 x-\sin^7 x$.

48. $\dfrac{2(3\tan\varphi-10\tan^3\varphi+3\tan^5\varphi)}{1-15\tan^2\varphi+15\tan^4\varphi-\tan^6\varphi}$.

49. $\cos nx=\cos^n x-C_n^2\cos^{n-2}x\sin^2 x+C_n^4\cos^{n-4}x\sin^4 x-\ldots+M$

where $M=(-1)^{\frac{n}{2}}\sin^n x$ if n is even, and

$M=(-1)^{\frac{n-1}{2}}\, n\cos x\sin^{n-1}x$ if n is odd.

$$\sin nx = C_n^1 \cos^{n-1} x \sin x - C_n^3 \cos^{n-3} x \sin^3 x + \ldots + M$$

where $M = (-1)^{\frac{n-2}{2}} n \cos x \sin^{n-1} x$ if n is even, and

$M = (-1)^{\frac{n-1}{2}} \sin^n x$ if n is odd.

50. (a) *Solution.* Let $\alpha = \cos x + i \sin x$. Then

$$\alpha^{-1} = \cos x - i \sin x;$$

$\alpha^k = \cos kx + i \sin kx$; $\alpha^{-k} = \cos kx - i \sin kx$.

Whence we have $\cos kx = \dfrac{\alpha^k + \alpha^{-k}}{2}$; $\sin kx = \dfrac{\alpha^k - \alpha^{-k}}{2i}$.

In particular, $\cos x = \dfrac{\alpha + \alpha^{-1}}{2}$; $\sin x = \dfrac{\alpha - \alpha^{-1}}{2i}$;

$$\sin^3 x = \left(\frac{\alpha - \alpha^{-1}}{2i}\right)^3 = \frac{\alpha^3 - 3\alpha + 3\alpha^{-1} - \alpha^{-3}}{-8i} = \frac{(\alpha^3 - \alpha^{-3}) - 3(\alpha - \alpha^{-1})}{-8i};$$

$$\sin^3 x = \frac{2i \sin 3x - 6i \sin x}{-8i} = \frac{3 \sin x - \sin 3x}{4};$$

(b) $\dfrac{\cos 4x - 4\cos 2x + 3}{8}$; (c) $\dfrac{\cos 5x + 5\cos 3x + 10\cos x}{16}$;

(d) $\dfrac{\cos 6x + 6\cos 4x + 15\cos 2x + 10}{32}$.

52. *Solution.*

$$C_{m-p}^p + C_{m-p-1}^{p-1} = \frac{(m-p)(m-p-1)\ldots(m-2p+1)}{p!} + \frac{(m-p-1)\ldots(m-2p+1)}{(p-1)!} = \frac{m(m-p-1)(m-p-2)\ldots(m-2p+1)}{p!}.$$

Denote $2\cos mx = S_m$; $2\cos x = a$. Then the equation that interests us may be written thus:

$$S_m = a^m - ma^{m-2} + (C_{m-2}^2 + C_{m-3}^1)\, a^{m-4} - \ldots + (-1)^p (C_{m-p}^p + C_{m-p-1}^{p-1})\, a^{m-2p} + \ldots$$

It is easy to show that

$2\cos mx = 2\cos x \cdot 2\cos (m-1)x - 2\cos (m-2)x$ or, in our notations, $S_m = aS_{m-1} - S_{m-2}$.

It can readily be verified that for $m=1$ and $m=2$, the equation being proved holds true. Let us assume that

$$S_{m-1} = a^{m-1} - (m-1)\, a^{m-3} + (C_{m-3}^2 + C_{m-4}^1)\, a^{m-5} + \ldots + (-1)^p (C_{m-p-1}^p + C_{m-p-2}^{p-1})\, a^{m-2p-1} + \ldots$$

$$S_{m-2}=a^{m-2}-(m-2)a^{m-4}+(C_{m-4}^{2}+C_{m-5}^{1})\,a^{m-6}$$

$$+\ \dots\ +(-1)^{p-1}\,(C_{m-p-1}^{p-1}+C_{m-p-2}^{p-2})\,a^{m-2p}+\dots$$

Then $S_m=a^m-ma^{m-2}$

$$+\dots+(-1)^p\,(C_{m-p-1}^{p}+C_{m-p-2}^{p-1}+C_{m-p-1}^{p-1}+C_{m-p-2}^{p-2})\,a^{m-2p}+\dots$$

Bearing in mind that $C_n^k=C_{n-1}^k+C_{n-1}^{k-1}$, we get the required result.

53. $\dfrac{\sin mx}{\sin x}=(2\cos x)^{m-1}-C_{m-2}^{1}(2\cos x)^{m-3}$

$$+C_{m-3}^{2}(2\cos x)^{m-5}-\dots+(-1)^p\,C_{m-p-1}^{p}(2\cos x)^{m-2p-1}+\dots$$

54. (a) $2^{\frac{n}{2}}\cos\dfrac{n\pi}{4}$, (b) $2^{\frac{n}{2}}\sin\dfrac{n\pi}{4}$. **56.** $\dfrac{2^n}{3^{\frac{n-1}{2}}}\sin\dfrac{n\pi}{6}$.

59. (a) *Solution.*

$S=1+a\cos\varphi+a^2\cos 2\varphi+\dots+a^k\cos k\varphi.$

Form $T=a\sin\varphi+a^2\sin 2\varphi+\dots+a^k\sin k\varphi$;

$$S+Ti=1+a(\cos\varphi+i\sin\varphi)+a^2(\cos 2\varphi+i\sin 2\varphi)$$
$$+\dots+a^k(\cos k\varphi+i\sin k\varphi).$$

Setting $\alpha=\cos\varphi+i\sin\varphi$, we have

$$S+Ti=1+a\alpha+a^2\alpha^2+\dots+a^k\alpha^k=\frac{a^{k+1}\alpha^{k+1}-1}{a\alpha-1}.$$

S is equal to the real part of the sum obtained. We have

$$S+Ti=\frac{a^{k+1}\alpha^{k+1}-1}{a\alpha-1}\cdot\frac{a\alpha^{-1}-1}{a\alpha^{-1}-1}=\frac{a^{k+2}\alpha^{k}-a^{k+1}\alpha^{k+1}-a\alpha^{-1}+1}{a^2-a(\alpha+\alpha^{-1})+1}.$$

Whence $S=\dfrac{a^{k+2}\cos k\varphi-a^{k+1}\cos(k+1)\varphi-a\cos\varphi+1}{a^2-2a\cos\varphi+1}$.

(b) $\dfrac{a^{k+2}\sin(\varphi+kh)-a^{k+1}\sin[\varphi+(k+1)h]-a\sin(\varphi-h)+\sin\varphi}{a^2-2a\cos h+1}$.

(c) $\dfrac{\sin\dfrac{2n+1}{2}x}{2\sin\dfrac{x}{2}}$.

60. *Solution.*

$$T=\sin x+\sin 2x+\dots+\sin nx;$$

$$S=\cos x+\cos 2x+\dots+\cos nx.$$

Let $\alpha=\cos\frac{x}{2}+i\sin\frac{x}{2}$. Then $S+Ti=\alpha^2+\alpha^4+\ldots+\alpha^{2n}$,

$$S+Ti=\alpha^2\frac{\alpha^{2n}-1}{\alpha^2-1}=\alpha^2\frac{\alpha^n(\alpha^n-\alpha^{-n})}{\alpha(\alpha-\alpha^{-1})}$$

$$=\left(\cos\frac{n+1}{2}x+i\sin\frac{n+1}{2}x\right)\frac{\sin\frac{n}{2}x}{\sin\frac{x}{2}}.$$

Whence $T=\sin\frac{n+1}{2}x\dfrac{\sin\frac{nx}{2}}{\sin\frac{x}{2}}$.

61. $\dfrac{2(2-\cos x)}{5-4\cos x}$.

64. (a) $\dfrac{\sin\left(a+\frac{n-1}{2}h\right)\sin\frac{nh}{2}}{\cos\frac{h}{2}}$, if n is even,

$\dfrac{\cos\left(a+\frac{n-1}{2}h\right)\cos\frac{nh}{2}}{\cos\frac{h}{2}}$, if n is odd;

(b) $\dfrac{\cos\left(a+\frac{n-1}{2}h\right)\sin\frac{nh}{2}}{\cos\frac{h}{2}}$, if n is even ,

$\dfrac{\sin\left(a+\frac{n-1}{2}h\right)\cos\frac{nh}{2}}{\cos\frac{h}{2}}$, if n is odd.

66. (a) $2^n\cos^n\frac{x}{2}\cos\frac{n+2}{2}x$; (b) $2^n\cos^n\frac{x}{2}\sin\frac{n+2}{2}x$.

67. (a) $2^n\sin^n\frac{x}{2}\cos\frac{n\pi-(n+2)x}{2}$; (b) $2^n\sin^n\frac{x}{2}\sin\frac{(n+2)x-n\pi}{2}$.

68. The limit of the sum is equal to the vector depicting the number $\frac{3+i}{5}$.

69. $\dfrac{n}{2}-\dfrac{\sin 4nx}{4\sin 2x}$.

71. (a) $\dfrac{3\cos\dfrac{n+1}{2}x\sin\dfrac{nx}{2}}{4\sin\dfrac{x}{2}}+\dfrac{\cos\dfrac{3(n+1)}{2}x\sin\dfrac{3nx}{2}}{4\sin\dfrac{3x}{2}}$;

(b) $\dfrac{3\sin\dfrac{n+1}{2}x\sin\dfrac{nx}{2}}{4\sin\dfrac{x}{2}}-\dfrac{\sin\dfrac{3(n+1)}{2}x\sin\dfrac{3nx}{2}}{4\sin\dfrac{3x}{2}}$.

72. (a) $\dfrac{(n+1)\cos nx-n\cos(n+1)x-1}{4\sin^2\dfrac{x}{2}}$;

(b) $\dfrac{(n+1)\sin nx-n\sin(n+1)x}{4\sin^2\dfrac{x}{2}}$.

73. $e^a(\cos b+i\sin b)$.

75. (a) $-3,\ \dfrac{3\pm i\sqrt{3}}{2}$; (b) $-3,\ \dfrac{3\pm 5i\sqrt{3}}{2}$;

(c) $-7,\ -1\pm i\sqrt{3}$; (d) $-1,\ \dfrac{-5\pm 5i\sqrt{3}}{2}$; (e) $2,\ -1\pm\sqrt{3}$;

(f) $\sqrt[3]{2}-\sqrt[3]{4},\ \dfrac{\sqrt[3]{4}-\sqrt[3]{2}}{2}\pm\dfrac{i\sqrt{3}}{2}(\sqrt[3]{4}+\sqrt[3]{2})$;

(g) $\sqrt[3]{9}-2\sqrt[3]{3},\ \dfrac{2\sqrt[3]{3}-\sqrt[3]{9}}{2}\pm\dfrac{i\sqrt{3}}{2}(\sqrt[3]{9}+2\sqrt[3]{3})$;

(h) $1-\sqrt[3]{2}-\sqrt[3]{4},\ \dfrac{2+\sqrt[3]{2}+\sqrt[3]{4}}{2}\pm\dfrac{i\sqrt{3}}{2}(\sqrt[3]{4}-\sqrt[3]{2})$;

(i) $-(1+\sqrt[3]{3}+\sqrt[3]{9})$,

$\dfrac{-2+\sqrt[3]{3}+\sqrt[3]{9}}{2}\pm\dfrac{i\sqrt{3}}{2}(\sqrt[3]{9}-\sqrt[3]{3})$;

(j) $2,\ -1\pm 2i\sqrt{3}$; (k) $2,\ -1\pm 3i\sqrt{3}$; (l) $2,\ -1\pm 4i\sqrt{3}$;
(m) $1,\ -2\pm\sqrt{3}$; (n) $4,\ -1\pm 4i\sqrt{3}$; (o) $-2i,\ i,\ i$;
(p) $-1-i,\ -1-i,\ 2+2i$;

(q) $-(a+b),\ \dfrac{a+b}{2}\pm\dfrac{i\sqrt{3}}{2}(a-b)$;

(r) $-(a\sqrt[3]{f^2g}+b\sqrt[3]{fg^2})$,

$\dfrac{a\sqrt[3]{f^2g}+b\sqrt[3]{fg^2}}{2}\pm\dfrac{i\sqrt{3}}{2}(a\sqrt[3]{f^2g}-b\sqrt[3]{fg^2})$;

(s) 2, 1149, −0,2541, −1, 8608;

(t) 1,5981, 0,5115, −2,1007.

76. *Solution.*

$x_1-x_2=\alpha(1-\omega)+\beta(1-\omega^2)=(1-\omega)(\alpha-\beta\omega^2)$;

$x_1-x_3=\alpha(1-\omega^2)+\beta(1-\omega)=(1-\omega^2)(\alpha-\beta\omega)$;

$x_2-x_3=\alpha(\omega-\omega^2)+\beta(\omega^2-\omega)=(\omega-\omega^2)(\alpha-\beta)$;

$(x_1-x_2)(x_1-x_3)(x_2-x_3)=3(\omega-\omega^2)(\alpha^3-\beta^3)$;

$(x_1-x_2)^2(x_1-x_3)^2(x_3-x_2)^2=-27[(\alpha^3+\beta^3)^2-4\alpha^3\beta^3]=-27q^2-4p^3$.

77. *Solution.*

The cubic equation mentioned in "Hints to Solutions" $z^3-3(px+q)z+$ $+x^3+p^3-3qx-3pq=0$ having the obvious root $z=-(x+p)$. The other roots of this equation are $z_{2,3}=\dfrac{x+p\pm\sqrt{-3(x-p)^2+12q}}{2}$. By virtue of Problem 76, the left member of the equation under study may be given as

$$-\frac{1}{27}(z_2-z_3)^2(z_3-z_1)^2(z_1-z_2)^2$$

$$=-\frac{1}{27}[-3(x-p)^2+12q]\left[\frac{3(x+p)+\sqrt{-3(x-p)^2+12q}}{2}\right]^2$$

$$\times\left[\frac{3(x+p)-\sqrt{-3(x-p)^2+12q}}{2}\right]^2$$

$$=[(x-p)^2-4q](x^2+px+p^2-q)$$

whence the roots are readily found:

$$x_{1,2}=p\pm2\sqrt{q},\quad x_3=x_4=\frac{-p+\sqrt{4q-3p^2}}{2};$$

$$x_5=x_6=\frac{-p-\sqrt{4q-3p^2}}{2}.$$

78. The left member will be represented in the form

$\alpha^5+\beta^5+5(\alpha+\beta)(\alpha^2+\alpha\beta+\beta^2-a)(\alpha\beta-a)-2b=0$.

Answer. $x=\alpha+\beta$ where

$$\alpha=\sqrt[5]{b+\sqrt{b^2-a^5}},\quad \beta=\sqrt[5]{b-\sqrt{b^2-a^5}};\quad \alpha\beta=a.$$

79. (a) $\pm\sqrt{2}$, $1\pm i\sqrt{3}$; (b) $-1\pm\sqrt{6}$, $\pm i\sqrt{3}$;

(c) $\pm\sqrt{2}$, $\dfrac{1\pm i\sqrt{3}}{2}$; (d) $\dfrac{1\pm\sqrt{5}}{2}$, $\dfrac{3\pm\sqrt{5}}{2}$;

(e) $\dfrac{1\pm\sqrt{13}}{2}$, $1\pm i$; (f) $\dfrac{1\pm\sqrt{29}}{2}$, $\dfrac{5\pm i\sqrt{7}}{2}$;

(g) $\pm i,\ 1\pm i\sqrt{2}$; (h) $\pm\sqrt{5},\ \frac{1\pm i\sqrt{7}}{2}$;

(i) $\pm i,\ -1\pm i\sqrt{6}$; (j) $-2\pm 2\sqrt{2},\ -1\pm i$;

(k) $1,\ 3,\ 1\pm\sqrt{2}$; (l) $1,\ -1,\ 1\pm 2i$;

(m) $\frac{1+\sqrt{5}\pm\sqrt{22+2\sqrt{5}}}{4}$, $\frac{1-\sqrt{5}\pm\sqrt{22-2\sqrt{5}}}{4}$;

(n) $\frac{1+\sqrt{5}\pm\sqrt{30-6\sqrt{5}}}{4}$, $\frac{1-\sqrt{5}\pm\sqrt{30+6\sqrt{5}}}{4}$;

(o) $\frac{1+\sqrt{2}}{2}\pm\frac{1}{2}\sqrt{-1-2\sqrt{2}}$, $\frac{1-\sqrt{2}}{2}\pm\frac{1}{2}\sqrt{-1+2\sqrt{2}}$;

(p) $1+\sqrt{7}\pm\sqrt{6+2\sqrt{7}}$, $1-\sqrt{7}\pm\sqrt{6-2\sqrt{7}}$;

(q) $\frac{1\pm\sqrt{4\sqrt{3}-3}}{2}$, $\frac{1\pm\sqrt{-4\sqrt{3}-3}}{2}$;

(r) $\frac{1+\sqrt{5}\pm\sqrt{-2-6\sqrt{5}}}{4}$, $\frac{1-\sqrt{5}\pm\sqrt{-2+6\sqrt{5}}}{4}$;

(s) $\frac{1+\sqrt{2}\pm\sqrt{-5+2\sqrt{2}}}{4}$, $\frac{1-\sqrt{2}\pm\sqrt{-5-2\sqrt{2}}}{4}$;

(t) $\frac{1+\sqrt{3}\pm\sqrt{12+2\sqrt{3}}}{4}$, $\frac{1-\sqrt{3}\pm\sqrt{12-2\sqrt{3}}}{4}$.

80. *Solution.*

$$x^4+ax^3+bx^2+cx+d$$
$$=\left(x^2+\frac{a}{2}x+\frac{\lambda}{2}+mx+n\right)\left(x^2+\frac{a}{2}x+\frac{\lambda}{2}-mx-n\right);$$

$x_1x_2=\frac{\lambda}{2}+n$; $x_3x_4=\frac{\lambda}{2}-n$; $\lambda=x_1x_2+x_3x_4$.

81. (a) ± 1; (b) $1,\ -\frac{1}{2}\pm i\frac{\sqrt{3}}{2}$; (c) $\pm 1,\ \pm i$;

(d) $\pm 1,\ \pm\frac{1}{2}\pm i\frac{\sqrt{3}}{2}$; (e) $\pm 1,\ \pm i,\ \pm\frac{\sqrt{2}}{2}(1\pm i)$;

(f) $\pm 1,\ \pm i,\ \pm\frac{1}{2}\pm i\frac{\sqrt{3}}{2},\ \pm\frac{\sqrt{3}}{2}\pm\frac{i}{2}$;

(g) $\pm 1,\ \pm i,\ \pm\frac{1}{2}\pm i\frac{\sqrt{3}}{2},\ \pm\frac{\sqrt{2}}{2}(1\pm i),\ \pm\frac{\sqrt{3}}{2}\pm\frac{i}{2}$,

$\pm\frac{\sqrt{6}+\sqrt{2}}{4}\pm i\frac{\sqrt{6}-\sqrt{2}}{4},\ \pm\frac{\sqrt{6}-\sqrt{2}}{4}\pm i\frac{\sqrt{6}+\sqrt{2}}{4}$.

82. (a) -1; (b) $-\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$; (c) $\pm i$; (d) $\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$;

(e) $\pm\frac{\sqrt{2}}{2}(1 \pm i)$; (f) $\pm\frac{\sqrt{3}}{2} \pm \frac{i}{2}$;

(g) $\pm\frac{\sqrt{6}+\sqrt{2}}{4} \pm i\frac{\sqrt{6}-\sqrt{2}}{4}$, $\pm\frac{\sqrt{6}-\sqrt{2}}{4} \pm i\frac{\sqrt{6}+\sqrt{2}}{4}$.

83. (a) 20, 20, 180; (b) 72, 144, 12.

84. $\cos\frac{2k\pi}{7} + i\sin\frac{2k\pi}{7}$, where $k=1, 2, 3, 4, 5, 6$.

85. (a) Denoting $\varepsilon_k = \cos\frac{2k\pi}{16} + i\sin\frac{2k\pi}{16}$, we get the following:

ε_0 belongs to exponent 1,
ε_8 belongs to exponent 2,
ε_4, ε_{12} belong to exponent 4,
ε_2, ε_6, ε_{10}, ε_{14} belong to exponent 8,
the primitive 16th roots are ε_1, ε_3, ε_5, ε_7, ε_9, ε_{11}, ε_{13}, ε_{15}.

(b) Denoting $\varepsilon_k = \cos\frac{2k\pi}{20} + i\sin\frac{2k\pi}{20}$, we find that

ε_0 belongs to exponent 1,
ε_{10} belongs to exponent 2,
ε_5, ε_{15} belong to exponent 4,
ε_4, ε_8, ε_{12}, ε_{16} belong to exponent 5,
ε_2, ε_6, ε_{14}, ε_{18} belong to exponent 10,
the primitive 20th roots are ε_1, ε_3, ε_7, ε_9, ε_{11}, ε_{13}, ε_{17}, ε_{19}.

(c) Denoting $\varepsilon_k = \cos\frac{2k\pi}{24} + i\sin\frac{2k\pi}{24}$, we find that

ε_0 belongs to exponent 1,
ε_{12} belongs to exponent 2,
ε_8, ε_{16} belong to exponent 3,
ε_6, ε_{18} belong to exponent 4,
ε_4, ε_{20} belong to exponent 6,
ε_3, ε_9, ε_{15}, ε_{21} belong to exponent 8,
ε_2, ε_{10}, ε_{14}, ε_{22} belong to exponent 12,
the primitive 24th roots are ε_1, ε_5, ε_7, ε_{11}, ε_{12}, ε_{17}, ε_{19}, ε_{23}.

86. (a) $X_1(x) = x-1$; (b) $X_2(x) = x+1$;

(c) $X_3(x) = x^2+x+1$; (d) $X_4(x) = x^2+1$;

(e) $X_5(x) = x^4+x^3+x^2+x+1$; (f) $X_6(x) = x^2-x+1$;

(g) $X_7(x) = x^6+x^5+x^4+x^3+x^2+x+1$; (h) $X_8(x) = x^4+1$;

(i) $X_9(x) = x^6+x^3+1$; (j) $X_{10}(x) = x^4-x^3+x^2-x+1$;

(k) $X_{11}(x) = x^{10}+x^9+x^8+x^7+x^6+x^5+x^4+x^3+x^2+x+1$;

(l) $X_{12}(x)=x^4-x^2+1$; (m) $X_{15}(x)=x^8-x^7+x^5-x^4+x^3-x+1$;

(n) $X_{105}(x)=x^{48}+x^{47}+x^{46}-x^{43}-x^{42}-2x^{41}-x^{40}-x^{39}+x^{36}+x^{35}$
$+x^{34}+x^{33}+x^{32}+x^{31}-x^{28}-x^{26}$
$-x^{24}-x^{22}-x^{20}+x^{17}+x^{16}+x^{15}+x^{14}+x^{13}+x^{12}-x^9-x^8$
$-2x^7-x^6-x^5+x^2+x+1$.

87. $\dfrac{2}{1-\varepsilon}$.

88. 0 if $n>1$.

89. n if k is divisible by n; 0 if k is not divisible by n.

90. $m(x^m+1)$.

91. $-\dfrac{n}{1-\varepsilon}$ if $\varepsilon\neq 1$; $\dfrac{n(n+1)}{2}$ if $\varepsilon=1$.

92. $-\dfrac{n^2(1-\varepsilon)+2n}{(1-\varepsilon)^2}$ if $\varepsilon\neq 1$; $\dfrac{n(n+1)(2n+1)}{6}$ if $\varepsilon=1$.

93. (a) $-\dfrac{n}{2}$; (b) $-\dfrac{n}{2}\cot\dfrac{\pi}{n}$.

94. (a) 1, (b) 0, (c) -1.

95. $x_0=1$;

$$x_1=\cos\frac{2\pi}{5}+i\sin\frac{2\pi}{5}=\frac{\sqrt{5}-1}{4}+\frac{i}{4}\sqrt{10+2\sqrt{5}};$$

$$x_2=\cos\frac{4\pi}{5}+i\sin\frac{4\pi}{5}=-\frac{\sqrt{5}+1}{4}+\frac{i}{4}\sqrt{10-2\sqrt{5}};$$

$$x_3=\cos\frac{6\pi}{5}+i\sin\frac{6\pi}{5}=-\frac{\sqrt{5}+1}{4}-\frac{i}{4}\sqrt{10-2\sqrt{5}};$$

$$x_4=\cos\frac{8\pi}{5}+i\sin\frac{8\pi}{5}=\frac{\sqrt{5}-1}{4}-\frac{i}{4}\sqrt{10+2\sqrt{5}}.$$

96. $\sin 18^\circ=\dfrac{\sqrt{5}-1}{4}$, $\cos 18^\circ=\dfrac{\sqrt{10+2\sqrt{5}}}{4}$.

97. *Solution.* Divide both sides of the equation $x^6+x^5+x^4+x^3+x^2+x+1=$ $=0$ by x^3. A few simple manipulations yield

$$\left(x+\frac{1}{x}\right)^3+\left(x+\frac{1}{x}\right)^2-2\left(x+\frac{1}{x}\right)-1=0.$$

The equation $z^3+z^2-2z-1=0$ is satisfied by $z=2\cos\dfrac{4\pi}{7}=-2\sin\dfrac{\pi}{14}$. Whence $t=2\sin\dfrac{\pi}{14}$ satisfies the equation $t^3-t^2-2t+1=0$. The resulting equation is the simplest in the sense that any other equation with rational coefficients having a common root with it is of higher degree. The proof of this fact requires information from subsequent sections of the course.

98. *Solution.* Let $n=2m$, then the equation $x^n-1=0$ has two real roots, 1 and -1 and $2m-2$ complex roots. Here $\varepsilon_k=\cos\frac{2k\pi}{2m}+i\sin\frac{2k\pi}{2m}$ is associated with $\varepsilon_{2m-k}=\cos\frac{2(2m-k)\pi}{2m}+i\sin\frac{2(2m-k)\pi}{2m}$. We thus have

$$x^{2m}-1=(x^2-1)(x-\varepsilon_1)(x-\bar{\varepsilon}_1)(x-\varepsilon_2)(x-\bar{\varepsilon}_2)\ldots(x-\varepsilon_{m-1})(x-\bar{\varepsilon}_{m-1});$$

$$x^{2m}-1=(x^2-1)[x^2-(\varepsilon_1+\bar{\varepsilon}_1)x+1]\ldots[x^2-(\varepsilon_{m-1}+\bar{\varepsilon}_{m-1})x+1];$$

$$x^{2m}-1=(x^2-1)\prod_{k=1}^{m-1}\left(x^2-2x\cos\frac{k\pi}{m}+1\right).$$

If $n=2m+1$, then in analogous fashion we obtain

$$x^{2m+1}-1=(x-1)\prod_{k=1}^{m}\left(x^2-2x\cos\frac{2k\pi}{2m+1}+1\right).$$

99. *Solution.* (a) We have

$$\frac{x^{2m}-1}{x^2-1}=\prod_{k=1}^{m-1}\left(x^2-2x\cos\frac{k\pi}{m}+1\right).$$

Putting $x=1$, we get $m=2^{m-1}\prod_{k=1}^{m-1}\left(1-\cos\frac{k\pi}{m}\right)$

or $m=2^{2(m-1)}\prod_{k=1}^{m-1}\sin^2\frac{k\pi}{2m}$ and finally,

$$\frac{\sqrt{m}}{2^{m-1}}=\prod_{k=1}^{m-1}\sin\frac{k\pi}{2m}.$$

Formula (b) is obtained in similar fashion.

100. *Solution.* In the identity $x^n-1=\prod_{k=0}^{n-1}(x-\varepsilon_k)$

where $\varepsilon_k=\cos\frac{2k\pi}{n}+i\sin\frac{2k\pi}{n}$, put $x=-\frac{a}{b}$. We get

$$(-1)^n\frac{a^n}{b^n}-1=(-1)^n\prod_{k=0}^{n-1}\left(\frac{a}{b}+\varepsilon_k\right)\text{ and so on.}$$

101. Following the suggestion made in "Hints to Solutions", we get

$$\cos n\Theta + i\sin n\Theta - 1 = \prod_{k=0}^{n-1} (\cos\Theta + i\sin\Theta - \varepsilon_k),$$

$$\cos n\Theta - i\sin n\Theta - 1 = \prod_{k=0}^{n-1} (\cos\Theta - i\sin\Theta - \varepsilon_k).$$

We get the required result by multiplying together the last equations.

102. *Solution.*

$$\prod_{k=0}^{n-1} \frac{(t+\varepsilon_k)^n - 1}{t} = \frac{1}{t^n} \prod_{k=0}^{n-1}\prod_{s=0}^{n-1} (t+\varepsilon_k-\varepsilon_s)$$

$$= \frac{1}{t^n} \prod_{k=0}^{n-1}\prod_{s=0}^{n-1} \left[t-\varepsilon_k\left(\frac{\varepsilon_s}{\varepsilon_k}-1\right)\right] = \frac{1}{t^n} \prod_{k=0}^{n-1}\prod_{s=0}^{n-1} [t-\varepsilon_k(\varepsilon_s-1)]$$

$$= \frac{1}{t^n} \prod_{s=0}^{n-1}\prod_{k=0}^{n-1} [t-\varepsilon_k(\varepsilon_s-1)] = \frac{1}{t^n} \prod_{s=0}^{n-1} [t^n-(\varepsilon_s-1)^n]$$

$$= \prod_{k=1}^{n-1} [t^n-(\varepsilon_k-1)^n].$$

103. We have $|x|=|x|^{n-1}$, hence $|x|=0$ or $|x|=1$. If $|x|=0$, then $x=0$. But if $|x|=1$, then $x\bar{x}=1$.

On the other hand, $x\bar{x}=x^n$. Hence, $x^n=1$. Thus

$$x=0 \text{ and } x=\cos\frac{2k\pi}{n}+i\sin\frac{2k\pi}{n},\quad k=0, 1, 2, \ldots, n-1.$$

The converse is readily verified.

104. *Solution.* If z satisfies the given equation, then $\left|\frac{z-a}{z-b}\right| = \sqrt[n]{\left|\frac{\mu}{\lambda}\right|}$. The locus of points the distance from which to two given points is the given ratio is a circle (a straight line in a particular case).

105. (a) We have $\frac{x+1}{x-1} = \varepsilon_k$ where $\varepsilon_k = \cos\frac{2k\pi}{m} + i\sin\frac{2k\pi}{m}$, $k=1, 2, \ldots, m-1$. Whence $x=\frac{\varepsilon_k+1}{\varepsilon_k-1}$. Transformation of the last expression yields $x_k=i\cot\frac{k\pi}{m}$, $k=1, 2, \ldots, m-1$;

(b) $x_k=\cot\frac{k\pi}{m}$, $k=1, 2, \ldots, m-1$;

(c) $x_k = \dfrac{a}{\varepsilon_k \sqrt[n]{2} - 1}$

where $\varepsilon_k = \cos \dfrac{2k\pi}{n} + i \sin \dfrac{2k\pi}{n}$, $k = 0, 1, 2, \ldots, n-1$.

106. *Solution.* Let $A = \cos \varphi + i \sin \varphi$. Then
$\dfrac{1+ix}{1-ix} = \eta_k^2$ where $\eta_k = \cos \dfrac{\varphi + 2k\pi}{2m} + i \sin \dfrac{\varphi + 2k\pi}{2m}$, $k = 0, 1, \ldots, m-1$.
Whence,

$$x = \frac{\eta_k^2 - 1}{i(\eta_k^2 + 1)} = \frac{\eta_k - \eta_k^{-1}}{i(\eta_k + \eta_k^{-1})} = \tan \frac{\varphi + 2k\pi}{2m}.$$

107. *Solution.* Using the hint, we have

$S + Ti = \mu(1 + \lambda x)^n$, $S - Ti = \bar{\mu}(1 + \bar{\lambda} x)^n$

where $\lambda = \cos \alpha + i \sin \alpha$, $\mu = \cos \varphi + i \sin \varphi$. Whence

$$2S = \mu(1 + \lambda x)^n + \bar{\mu}(1 + \bar{\lambda} x)^n.$$

The equation becomes $\mu(1 + \lambda x)^n + \bar{\mu}(1 + \bar{\lambda} x)^n = 0$;

$$x_k = -\frac{\sin \dfrac{(2k+1)\pi - 2\varphi}{2n}}{\sin \dfrac{(2k+1)\pi - 2\varphi - 2n\alpha}{2n}}, \qquad k = 0, 1, 2, \ldots, n-1.$$

108. *Solution.* Let $\alpha^a = 1$, $\beta^b = 1$. Then $(\alpha\beta)^{ab} = (\alpha^a)^b \cdot (\beta^b)^a = 1$.

109. *Solution.* Let ε be the common root of $x^a - 1$ and $x^b - 1$; s is the exponent to which ε belongs. Then s is a common divisor of a and b and can therefore only be equal to 1 and $\varepsilon = 1$. The converse is obvious.

110. *Solution.* Let α_k and β_s be roots of 1 of degree a and b; $k = 0, 1, 2, \ldots, a-1$; $s = 0, 1, 2, \ldots, b-1$. On the basis of Problem 108 it suffices to show that all $\alpha_k \beta_s$ are distinct. Suppose that $\alpha_{k_1}\beta_{s_1} = \alpha_{k_2}\beta_{s_2}$. Then $\dfrac{\alpha_{k_1}}{\alpha_{k_2}} = \dfrac{\beta_{s_2}}{\beta_{s_1}}$, i.e., $\alpha_i = \beta_j$. On the basis of Problem 109, $\alpha_i = \beta_j = 1$, i.e., $k_1 = k_2$, $s_1 = s_2$.

111. *Solution.* Let α and β be primitive ath and bth roots of 1. Let $(\alpha\beta)^s = 1$. Then $\alpha^{bs} = 1$, $\beta^{as} = 1$. It thus appears that bs is divisible by a and as is divisible by b. Hence, s is divisible by ab.

Let λ be a primitive abth root of unity. Then $\lambda = \alpha^k \beta^s$ (Problem 110). Let α^k belong to the exponent $a_1 < a$. Then $\lambda^{a_1 b} = (\alpha^k)^{a_1 b} (\beta^s)^{a_1 b} = 1$ which is impossible. In the same way it may be shown that β^s is a primitive bth root of 1.

112. It follows directly from Problem 111.

113. Using the hint, write out all multiples of p that do not exceed p^α. Namely, $1 \cdot p$, $2 \cdot p$, $3 \cdot p$, $\ldots$, $p^{\alpha-1} \cdot p$. It is immediately seen that there are $p^{\alpha-1}$ such numbers. Whence $\varphi(p^\alpha) = p^\alpha - p^{\alpha-1} = p^\alpha\left(1 - \dfrac{1}{p}\right)$. On the basis of Problem 112, $\varphi(n)=\varphi(p_1^{\alpha_1})\varphi(p_2^{\alpha_2})\ldots\varphi(p_k^{\alpha_k})=n\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\ldots\left(1-\frac{1}{p_k}\right)$.

114. *Solution.* If ε is a primitive nth root of 1, then $\bar{\varepsilon}$, the conjugate of ε, is also a primitive nth root of 1. Here, $\varepsilon\neq\pm 1$ since $n>2$.

115. $X_p(x)=x^{p-1}+x^{p-2}+\ldots+x+1$.

116. $X_{p^m}(x)=x^{(p-1)p^{m-1}}+x^{(p-2)p^{m-1}}+\ldots+x^{p^{m-1}}+1$.

117. The suggestion can be utilized immediately on the basis of Problem 111.

Let $\alpha_1, \alpha_2, \ldots, \alpha_{\varphi(n)}$ be primitive nth roots of 1. Then $-\alpha_1, -\alpha_2, \ldots, -\alpha_{\varphi(n)}$ are primitive $2n$th roots of 1. We have

$X_{2n}(x)=(x+\alpha_1)(x+\alpha_2)\ldots(x+\alpha_{\varphi(n)})=(-1)^{\varphi(n)}(-x-\alpha_1)\ldots(-x-\alpha_{\varphi(n)})$ or (Problem 114) $X_{2n}(x)=X_n(-x)$.

118. *Solution.* Let $\varepsilon_k=\cos\frac{2k\pi}{nd}+i\sin\frac{2k\pi}{nd}$ be a primitive ndth root of 1, that is, k and n are relatively prime. Divide k by n, this yields $k=nq+r$, where $0<r<n$. Whence $\varepsilon_k=\cos\frac{2q\pi+\frac{2r\pi}{n}}{d}+i\sin\frac{2q\pi+\frac{2r\pi}{n}}{d}$ that is, ε_k is one of the values of the dth root of $\eta_r=\cos\frac{2r\pi}{n}+i\sin\frac{2r\pi}{n}$; η_r is a primitive nth root of 1, since every common divisor of r and n is a common divisor of k and n.

Now let $\eta_r=\cos\frac{2r\pi}{n}+i\sin\frac{2r\pi}{n}$ be a primitive nth root of 1, i.e., r and n are relatively prime.

Form $\varepsilon_q=\cos\frac{2q\pi+\frac{2r\pi}{n}}{d}+i\sin\frac{2q\pi+\frac{2r\pi}{n}}{d}=\cos\frac{2\pi(r+nq)}{nd}+$ $+i\sin\frac{2\pi(r+nq)}{nd}$ where $q=0, 1, 2, \ldots, d-1$; ε_q is a primitive ndth root of 1. Indeed, if $r+nq$ and nd were both divisible by some prime p, then p would divide n and r, but this is impossible.

119. *Solution.* Let $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{\varphi(n')}$ be primitive roots of 1 of degree n'. Then $X_{n'}(x^{n''})=\prod_{k=1}^{\varphi(n')}(x^{n''}-\varepsilon_k)$. Furthermore, let $(x-\varepsilon_{k,1})(x-\varepsilon_{k,2})\ldots(x-\varepsilon_{k,n''})$ be a factorization of $x^{n''}-\varepsilon_k$ into linear factors. Then $X'_n(x^{n''})=$ $=\prod_{\substack{k=1\\ i=1}}^{\substack{k=\varphi(n')\\ i=n''}}(x-\varepsilon_{k,i})$. On the basis of Problem 118, each linear factor $x-\varepsilon_{k,i}$

enters into the factorization of $X_n(x)$, and conversely. Besides, since $\varphi(n)=n''\varphi(n')$, the degrees of $X_n(x)$ and $X_{n'}(x^{n''})$ are equal.

121. *Solution.* The sum of all nth roots of 1 is zero. Since each nth root of 1 belongs to the exponent d, which is a divisor of n, and conversely, it follows that $\sum\limits_{d/n} \mu(d)=0$.

122. *Solution.* Let $\varepsilon_k=\cos\frac{2k\pi}{n}+i\sin\frac{2k\pi}{n}$ belong to the exponent n_1. Then the factor $x-\varepsilon_k$ will enter into the binomials x^d-1, where d is divisible by n_1 (and only such binomials). Here, if d runs through all divisors of n that are multiples of n_1, $\frac{n}{d}$ runs through all divisors of $\frac{n}{n_1}$. Thus, $x-\varepsilon_k$ will enter the right side with exponent $\sum\limits_{d/\frac{n}{n_1}} \mu(d_1)$. This sum is equal to zero if $\frac{n}{n_1}\neq 1$ and to 1 if $n=n_1$.

123. *Solution.* If $n=p^\alpha$ where p is prime, then $X_n(1)=p$. If $n=p_1^{\alpha_1}p_2^{\alpha_2}\dots p_k^{\alpha_k}$, where $p_1, p_2, \dots, p_k$ are distinct primes, then (Problem 119) $X_n(1)=X_{n'}(1)$, where $n'=p_1, p_2\dots p_k$.

Now let $n=p_1, p_2\dots p_k$; $k\geqslant 2$; $n_1=\frac{n}{p_k}$. Note that in order to obtain all divisors of n it is sufficient to adjoin to all divisors of n_1 their products into p_k. Therefore

$$X_n(x)=\prod_{d/n}(x^d-1)^{\mu\left(\frac{n}{d}\right)}$$

$$=\prod_{d/n_1}(x^d-1)^{\mu\left(\frac{n}{d}\right)}\cdot\prod_{d/n_1}(x^{dp_k}-1)^{\mu\left(\frac{n}{dp_k}\right)}$$

$$=[X_{n_1}(x)]^{-1}\cdot X_{n_1}(x^{p_k}).$$

Whence $X_n(1)=1$.

124. *Solution.* (1) Let n be odd and greater than unity. Then (Problem 117) $X_n(-1)=X_{2n}(1)=1$.

(2) Let $n=2^k$, then $X_n=\frac{x^n-1}{x^{\frac{n}{2}}-1}=x^{\frac{n}{2}}+1$ and $X_n(-1)$ is equal to 0 if $k=1$ and to 2 if $k>1$.

(3) Let $n=2n_1$ where n_1 is odd and greater than unity. Then (Problem 117) $X_n(-1)=X_{n_1}(1)$ and, hence, $X_n(-1)$ is equal to p if $n_1=p^\alpha$ (p prime) or is equal to 1 if $n_1\neq p^\alpha$.

(4) Let $n=2^k n_1$, where $k>1$ and $n_1=p_1^{\alpha_1}p_2^{\alpha_2}\dots p_s^{\alpha_s}$ ($p_1, p_2, \dots, p_s$ are distinct odd primes). In this case (Problem 119) $X_n(x)=X_{2p_1p_2\dots p_s}(x^\lambda)$, where $\lambda=2^{k-1}p_1^{\alpha_1-1}\dots p_s^{\alpha_s-1}$. Whence it follows that $X_n(-1)=X_n(1)=1$.

125. *Solution.* Let $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{\varphi(n)}$ be primitive nth roots of 1:

$$s=\varepsilon_1\varepsilon_2+\varepsilon_1\varepsilon_3+\ldots+\varepsilon_{\varphi(n)-1}\cdot\varepsilon_{\varphi(n)}=\frac{[\mu(n)]^2-(\varepsilon_1^2+\varepsilon_2^2+\ldots+\varepsilon_{\varphi(n)}^2)}{2}.$$

(1) Let n be odd, then ε_i^2 is a primitive nth root of 1 and $\varepsilon_i^2=\varepsilon_j^2$ for $i=j$ only. Therefore

$$\varepsilon_1^2+\varepsilon_2^2+\ldots+\varepsilon_{\varphi(n)}^2=\mu(n) \quad \text{and} \quad s=\frac{[\mu(n)]^2-\mu(n)}{2}.$$

(2) Let $n=2n_1$, n_1 is odd. In this case $-\varepsilon_i$ (Problem 111) is a primitive n_1th root of unity and therefore [see (1)] $\varepsilon_1^2+\varepsilon_2^2+\ldots+\varepsilon_{\varphi(n)}^2=\mu(n_1)=-\mu(n)$. Thus, in this case $s=\dfrac{[\mu(n)]^2+\mu(n)}{2}$.

(3) Let $n=2^kn_1$ where $k>1$, n_1 is odd. In this case ε_i^2 belongs to exponent $\dfrac{n}{2}$. On the basis of Problem 118, we assert that $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{\varphi(n)}$ are square roots of $\eta_1, \eta_2, \ldots, \eta_{\varphi\left(\frac{n}{2}\right)}$, where $\eta_1, \eta_2, \ldots, \eta_{\varphi\left(\frac{n}{2}\right)}$ are primitive $\dfrac{n}{2}$ th roots of 1. Whence it follows that

$$\varepsilon_1^2+\varepsilon_2^2+\ldots+\varepsilon_{\varphi(n)}^2=2\left(\eta_1+\eta_2+\ldots+\eta_{\varphi\left(\frac{n}{2}\right)}\right)=2\mu\left(\frac{n}{2}\right);$$

$$s=-\mu\left(\frac{n}{2}\right).$$

126. *Solution.* $S=\sum\limits_{x=0}^{n-1}\varepsilon^{x^2}=\sum\limits_{x=y}^{y+n-1}\varepsilon^{x^2}=\sum\limits_{s=0}^{n-1}\varepsilon^{(y+s)^2}$

for arbitrary integral y:

$$S'=\sum_{y=0}^{n-1}\varepsilon^{-y^2};\quad S'S=\sum_{y=0}^{n-1}\varepsilon^{-y^2}S=\sum_{y=0}^{n-1}\left(\varepsilon^{-y^2}\cdot\sum_{s=0}^{n-1}\varepsilon^{(y+s)^2}\right)$$

$$=\sum_{y=0}^{n-1}\sum_{s=0}^{n-1}\varepsilon^{2ys+s^2}=\sum_{s=0}^{n-1}\left(\varepsilon^{s^2}\cdot\sum_{y=0}^{n-1}\varepsilon^{2ys}\right)=n+\sum_{s=1}^{n-1}\varepsilon^{s^2}$$

$$\cdot\sum_{y=0}^{n-1}(\varepsilon^{2s})^y=n \quad \text{for } n \text{ odd};$$

$$SS'=n+n\varepsilon^{\left(\frac{n}{2}\right)^2}=n\left[1+(-1)^{\frac{n}{2}}\right]$$

for n even (since $\sum\limits_{y=0}^{n-1}\varepsilon^{2sy}=0$ for $2s$ not divisible by n).

To summarize, $|S|=\sqrt{n}$ if n is odd, and $|S|=\sqrt{n\left[1+(-1)^{\frac{n}{2}}\right]}$ if n is even.

CHAPTER 2

EVALUATION OF DETERMINANTS

127. (a) 5, (b) 5, (c) 1, (d) $ab-c^2-d^2$, (e) $\alpha^2+\beta^2-\gamma^2-\delta^2$, (f) $\sin(\alpha-\beta)$, (g) $\cos(\alpha+\beta)$, (h) $\sec^2\alpha$, (i) -2, (j) 0, (k) $(b-c)(d-a)$, (l) $4ab$, (m) -1, (n) -1, (o) $\frac{1+i\sqrt{3}}{2}$.

128. (a) 1, (b) 2, (c) $2a^2(a+x)$, (d) 1, (e) -2, (f) $-2-\sqrt{2}$, (g) $-3i\sqrt{3}$, (h) -3.

129. The number of transpositions is odd.

130. (a) 10, (b) 18, (c) 36. **131.** (a) $i=8$, $k=3$, (b) $i=3$, $k=6$.

132. C_n^2. **133.** C_n^2-I. **134.** (a) $\frac{n(n-1)}{2}$, (b) $\frac{n(n+1)}{2}$.

135. (a) $\frac{n(3n+1)}{2}$, (b) $\frac{3n(n-1)}{2}$.

136. Consider a pair of elements a_i and a_k, where $i<k$. If these elements do not form an inversion, then a_i will precede a_k when the permutation returns to its original order, and, hence, the indices will not yield an inversion.

But if the elements a_i and a_k form an inversion, then a_k will precede a_i after the permutation is returned to its original ordering and, thus, the indices i and k will yield an inversion.

137. In both cases the permutation is odd. This is due to the fact that the original arrangement is obtained from the other one by means of an even number of transpositions.

138. (a) With the + sign; (b) with the + sign.

139. (a) No, (b) yes. **140.** $i=1$, $k=4$.

141. $a_{11}a_{23}a_{32}a_{44}$, $a_{12}a_{23}a_{34}a_{41}$ and $a_{14}a_{23}a_{31}a_{42}$.

142. $$-a_{14}a_{23}\begin{vmatrix} a_{31} & a_{32} & a_{35} \\ a_{41} & a_{42} & a_{45} \\ a_{51} & a_{52} & a_{55} \end{vmatrix}.$$

143. With the + sign. **144.** With the sign $(-1)^{C_n^2}$.

146. 2, -1. **147.** (a) $n!$, (b) $(-1)^{\frac{n(n-1)}{2}}$, (c) $n!$.

148. (a) $(-1)^{\frac{n(n+1)}{2}}(n!)^{n+1}$, (b) $(-1)^{\frac{n(n+1)}{2}}(n!)^{n+1}$.

149. *Solution.* Interchanging rows and columns: (1) does not change the determinant; (2) makes the determinant a conjugate complex number.

150. *Solution.* Interchanging rows and columns: (1) does not change the determinant; (2) results in the determinant being multiplied by -1.

151. $(-1)^{n-1}\Delta$. **152.** Is multiplied by $(-1)^{\frac{n(n-1)}{2}}$.

153. Zero, since the number of even permutations of n elements is equal to the number of their odd permutations.

154. (a) $x_1=a_1;\ x_2=a_2;\ \ldots;\ x_{n-1}=a_{n-1}$;
(b) $x_1=0;\ x_2=1;\ \ldots;\ x_{n-1}=n-2$;
(c) $x_1=a_1;\ x_2=a_2;\ \ldots;\ x_{n-1}=a_{n-1}$.

156. 0. **158.** $(mq-np)\begin{vmatrix} a & b \\ c & d \end{vmatrix}$.

159. (a) $a_1 a_2 \ldots a_n \left(\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_n}\right)$;

(b) $(-1)^{\frac{n(n-1)}{2}} a_1 a_2 \ldots a_n \left(\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_n}\right)$.

160. $3a-b+2c+d$. **161.** $4t-x-y-z$. **162.** $2a-b-c-d$.
163. $-1{,}487{,}600$. **164.** $-29{,}400{,}000$. **165.** 48. **166.** 1.
167. 160. **168.** 12. **169.** 900. **170.** 394. **171.** 665.
172. $a^2+b^2+c^2-2(bc+ca+ab)$. **173.** $-2(x^3+y^3)$.
174. $(x+1)(x^2-x+1)^2$. **175.** x^2z^2. **176.** $-3(x^2-1)(x^2-4)$.
177. $\sin(c-a)\sin(c-b)\sin(a-b)$. **178.** $(af-be+cd)^2$.
179. $n!$. **180.** $b_1b_2\ldots b_n$. **181.** $(x-x_1)(x-x_2)\ldots(x-x_n)$.
182. $(n-1)!$. **183.** $-2(n-2)!$. **184.** 1.

185. $\frac{na^{n-1}}{2}[2a+(n-1)h]$. **186.** $\frac{na^{n-1}}{2}[2a+(n-1)h]$.

187. $(-1)^{\frac{n(n+1)}{2}}[a_0-a_1+a_2-\ldots+(-1)^n a_n]$.

188. $a_0x^n+a_1x^{n-1}+\ldots+a_n$. **189.** $\frac{nx^n}{x-1}-\frac{x^n-1}{(x-1)^2}$.

190. $(n+1)!\,x^n$. **191.** $(x-a_1)(x-a_2)\ldots(x-a_n)$.

192. $[x+(n-1)a](x-a)^{n-1}$. **193.** $\frac{(x+a)^n+(x-a)^n}{2}$.

194. $(-1)^n(n+1)a_1a_2\ldots a_n$.

195. $a_1a_2\ldots a_n\left(1+\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_n}\right)$.

196. $h(x+h)^n$. **197.** $(-1)^{n-1}(n-1)x^{n-2}$.

198. $(-1)^n 2^{n-1} a_1a_2\ldots a_n\left(\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_n}\right)$.

199. $(-1)^{\frac{n(n-1)}{2}}\frac{n^{n-1}(n+1)}{2}$. **200.** $\frac{(n+1)^{n+1}}{n^n}$.

201. $\prod_{k=1}^{n}(1-ax_{k,k})$. **202.** $(-1)^{n-1}2^{n-2}(n+1)$.

203. $(-1)^n (a_0b_0+a_1b_1+\ldots+a_nb_n)\, b_1b_2\ldots b_{n-1}$.

204. $a(a+b)(a+2b)\ldots[a+(n+1)b]$.

205. $x^n+(-1)^{n-1}y^n$. **206.** 0 if $n>2$. **207.** 0 if $n>2$.

208. Let $n=2$. Using the hint, we have

$$\begin{vmatrix} 1+a_1+x_1 & a_1+x_2 \\ a_2+x_1 & 1+a_2+x_2 \end{vmatrix}$$

$$=\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}+\begin{vmatrix} 1 & a_1+x_2 \\ 0 & a_2+x_2 \end{vmatrix}+\begin{vmatrix} a_1+x_1 & 0 \\ a_2+x_1 & 1 \end{vmatrix}+\begin{vmatrix} a_1+x_1 & a_1+x_2 \\ a_2+x_1 & a_2+x_2 \end{vmatrix}$$

$$=1+[(a_1+x_1)+(a_2+x_2)]+(a_2-a_1)(x_1-x_2).$$

If in this fashion we represent an nth-order determinant in the form of a sum of 2^n determinants, we find that one of the summands is equal to unity, n summands are equal to a_i+x_i, where $i=1, 2, \ldots, n$ and $\frac{n(n-1)}{2}$ summands are equal to $(a_i-a_k)(x_k-x_i)$, where $i>k$.

The other summands are zero. Thus, we have the answer:

$$1+\sum_{i=1}^{n}(a_i+x_i)+\sum_{i>k}(a_i-a_k)(x_k-x_i).$$

This result can be transformed to

$$(1+a_1+a_2+\ldots+a_n)\,(1+x_1+x_2+\ldots+x_n)-n\,(a_1x_1+a_2x_2+\ldots+a_nx_n).$$

209. 0 if $p>2$. **211.** $\frac{n+1}{1-x}+\frac{x^{n+1}-1}{(1-x)^2}$.

212. *Solution.* It is easy to see that $\Delta_2=x_1x_2\left(1+\frac{a_1}{x_1}+\frac{a_2}{x_2}\right)$. Suppose that

$$\Delta_{n-1}=x_1x_2\ldots x_{n-1}\left(1+\frac{a_1}{x_1}+\frac{a_2}{x_2}+\ldots+\frac{a_{n-1}}{x_{n-1}}\right).$$

Then $\Delta_n=x_1x_2\ldots x_n\left(1+\frac{a_1}{x_1}+\frac{a_2}{x_2}+\ldots+\frac{a_{n-1}}{x_{n-1}}\right)$

$$+a_nx_1x_2\ldots x_{n-1}=x_1x_2\ldots x_n\left(1+\frac{a_1}{x_1}+\frac{a_2}{x_2}+\ldots+\frac{a_n}{x_n}\right).$$

213. $a_0x_1x_2\ldots x_n+a_1y_1x_2\ldots x_n+a_2y_1y_2x_3\ldots x_n+\ldots+a_ny_1y_2\ldots y_n$.

214. $-a_1a_2\ldots a_n\left(\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_n}\right)$.

215. $n!\,(a_0x^n+a_1x^{n-1}+\ldots+a_n)$.

216. $a_1a_2\ldots a_{n-1}-a_1a_2\ldots a_{n-2}+\ldots+(-1)^na_1+(-1)^{n+1}$.

217. *Solution.* Expand the determinant by elements of the first column; this yields $\Delta_n=(\alpha+\beta)\,\Delta_{n-1}-\alpha\beta\Delta_{n-2}$. It is easy to verify that

$\Delta_2=\frac{\alpha^3-\beta^3}{\alpha-\beta}$, $\Delta_3=\frac{\alpha^4-\beta^4}{\alpha-\beta}$. Suppose that $\Delta_{n-2}=\frac{\alpha^{n-1}-\beta^{n-1}}{\alpha-\beta}$, $\Delta_{n-1}=$ $=\frac{\alpha^n-\beta^n}{\alpha-\beta}$. Then

$$\Delta_n=(\alpha+\beta)\frac{\alpha^n-\beta^n}{\alpha-\beta}-\alpha\beta\frac{\alpha^{n-1}-\beta^{n-1}}{\alpha-\beta}=\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}.$$

Alternative solution.

Represent Δ_n as a sum $d_n+\delta_n$, where

$$d_n=\begin{vmatrix} \alpha & \alpha\beta & 0 & \dots & 0 & 0 \\ 1 & \alpha+\beta & \alpha\beta & \dots & 0 & 0 \\ 0 & 1 & \alpha+\beta & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & \alpha+\beta \end{vmatrix},$$

$$\delta_n=\begin{vmatrix} \beta & 0 & 0 & \dots & 0 & 0 \\ 1 & \alpha+\beta & \alpha\beta & \dots & 0 & 0 \\ 0 & 1 & \alpha+\beta & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & \alpha+\beta \end{vmatrix}.$$

Take α out of the first row of d_n and then subtract the first row from the second. This yields $d_n=\alpha d_{n-1}$. It is readily seen that $d_2=\alpha^2$. Let $d_{n-1}=\alpha^{n-1}$, then $d_n=\alpha^n$.

Expanding δ_n by elements of the first row, we see that

$$\delta_n=\beta\Delta_{n-1}.$$

From the foregoing it follows that $\Delta_n=\alpha^n+\beta\Delta_{n-1}$. We can readily check that $\Delta_2=\frac{\alpha^3-\beta^3}{\alpha-\beta}$. Suppose that $\Delta_{n-1}=\frac{\alpha^n-\beta^n}{\alpha-\beta}$. Then

$$\Delta_n=\alpha^n+\beta\frac{\alpha^n-\beta^n}{\alpha-\beta}=\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}.$$

218. $n+1$. **219.** $\frac{\sin(n+1)\Theta}{\sin\Theta}$. **220.** $\cos n\Theta$.

221. $x^n-C_{n-1}^1x^{n-2}+C_{n-2}^2x^{n-4}-\dots$ Compare with Problem 53.

222. $x_1y_n\prod_{i=1}^{n-1}(x_{i+1}y_i-x_iy_{i+1})$.

223. $a_1a_2\dots a_n\left(1+\frac{1}{a_1}+\frac{1}{a_2}+\dots+\frac{1}{a_n}\right)$.

224. $(-1)^{\frac{n(n-1)}{2}}a_1a_2\dots a_n\left(1+\frac{1}{a_1}+\frac{1}{a_2}+\dots+\frac{1}{a_n}\right)$.

225. $x(a_1-x)\dots(a_n-x)\left(\frac{1}{x}+\frac{1}{a_1-x}+\dots+\frac{1}{a_n-x}\right)$.

226. $(x_1-a_1)(x_2-a_2)\ldots(x_n-a_n)\left(1+\frac{a_1}{x_1-a_1}+\ldots+\frac{a_n}{x_n-a_n}\right).$

227. $\prod_{i=1}^{n}(x_i-a_ib_i)\left(1+\sum_{i=1}^{n}\frac{a_ib_i}{x_i-a_ib_i}\right).$

228. $(-1)^n m^n\left(1-\sum_{i=1}^{n}\frac{x_i}{m}\right).$

229. $x_1=x_2=\ldots=x_{n-1}=0;\quad x_n=\sum_{i=1}^{n}\frac{a_i}{\alpha_i}.$

230. $(a^2-b^2)^n.$

231. $a(a+b)\ldots[a+(n-1)b]\left(\frac{1}{a}+\frac{1}{a+b}+\ldots+\frac{1}{a+(n-1)b}\right).$

232. $x^{n-1}\prod_{i=1}^{n}(x-2a_i)\left(x+\sum_{i=1}^{n}\frac{a_i^2}{x-2a_i}\right).$

233. $x^{n-1}\prod_{i=1}^{n}(x-2a_i)\left(x+\sum_{i=1}^{n}\frac{a_i^2}{x-2a_i}\right).$

234. $1-b_1+b_1b_2-b_1b_2b_3+\ldots+(-1)^n b_1b_2\ldots b_n.$

235. $(-1)^{n-1}(b_1a_2a_3\ldots a_n+b_1b_2a_3\ldots a_n+\ldots+b_1b_2\ldots b_{n-1}a_n).$

236. $(-1)^{n-1}x^{n-2}.$ **237.** $(-1)^n[(x-1)^n-x^n].$

238. $a_0x^n\prod_{i=1}^{n}(b_i-a_i).$ **240.** 1. **241.** 1. **242.** 1.

243. $\dfrac{C_{m+n}^{n+1}C_{m+n-1}^{n+1}\ldots C_{m+n-k+1}^{n+1}}{C_{k+n}^{n+1}C_{k+n-1}^{n+1}\ldots C_{n+1}^{n+1}}.$ **244.** $(-1)^{\frac{m(m+1)}{2}}.$

245. $(x-1)^n.$ **246.** $(n-1)!\,(n-2)!\ldots1!\,(x-1)^n.$ **247.** $\alpha^n.$

248. Using the hint, we have

$$\Delta_n=(x-z)\Delta_{n-1}+z(x-y)^{n-1},$$

$$\Delta_n=(x-y)\Delta_{n-1}+y(x-z)^{n-1}.$$

From the resulting system of equations, we find

$$\Delta_n=\frac{z(x-y)^n-y(x-z)^n}{z-y}.$$

249. $(-1)^{\frac{n(n+1)}{2}}\dfrac{ab(b^{n-1}-a^{n-1})}{a-b}.$

250. $\dfrac{xf(y)-yf(x)}{x-y}$ where $f(x)=\prod\limits_{k=1}^{n}(a_k-x)$.

251. $\dfrac{f(a)-f(b)}{a-b}$ where $f(x)=\prod\limits_{k=1}^{n}(c_k-x)$.

252. $(\alpha-\beta)^{n-2}[\lambda\alpha+(n-2)\lambda\beta-(n-1)ab]$.

253. $(-1)^{\frac{n(n-1)}{2}}\dfrac{n^{n-1}(n+1)}{2}$.

254. $(-1)^{\frac{n(n-1)}{2}}(nh)^{n-1}\left[a+\dfrac{h(n-1)}{2}\right]$. **255.** $(1-x^n)^{n-1}$.

256. If all columns are added to the first column, then we can take $a+b+$ $+c+d$ outside the sign of the determinant; all elements of the remaining determinant will be integral expressions in a.

This proves that the determinant is divisible by $a+b+c+d$. If we add the second column to the first and subtract the third and fourth, we find that the determinant is divisible by $a+b-c-d$. Reasoning in this way, we will show that the determinant is divisible by $a-b+c-d$ and $a-b-c+d$. From the foregoing it follows that the determinant is equal to $\lambda(a+b+c+d)(a+b-$ $-c-d)$ $(a-b+c-d)$ $(a-b-c+d)$. To determine λ, note that the coeffi cient of a^4 must be equal to 1, and so $\lambda=1$.

257. $(a+b+c+d+e+f+g+h)$ $(a+b+c+d-e-f-g-h)\times$ $\times(a+b-c-d+e+f-g-h)$ $(a+b-c-d-e-f+g+h)$ $(a-b+c$ $-d+e-f+g-h)$ $(a-b+c-d-e+f-g+h)$ $(a-b-c+d+e-f$ $-g+h)$ $(a-b-c+d-e+f+g-h)$.

258. $(x+a_1+a_2+\ldots+a_n)$ $(x-a_1)$ $(x-a_2)\ldots(x-a_n)$.

259. $2^{\frac{n(n-1)}{2}}\prod\limits_{1\leqslant i<k\leqslant n}\sin\dfrac{\varphi_i+\varphi_k}{2}\prod\limits_{1\leqslant i<k\leqslant n}\sin\dfrac{\varphi_k-\varphi_i}{2}$.

260. $2^{\frac{n(n-1)}{2}}\prod\limits_{n\geqslant i>k\geqslant 1}\cos\dfrac{\varphi_i+\varphi_k}{2}\prod\limits_{n\geqslant i>k\geqslant 1}\sin\dfrac{\varphi_i-\varphi_k}{2}$.

261. $1!\,2!\ldots n!$ **262.** $\prod\limits_{n+1\geqslant k>i\geqslant 1}(a_i-a_k)$.

263. $(-1)^n\,1!\,2!\ldots n!$.

264. $(-1)^{n-1}\prod\limits_{i=1}^{n}a_i\prod\limits_{n\geqslant i>k\geqslant 1}(a_i-a_k)\left(\sum\limits_{i=1}^{n}\dfrac{w_i}{a_i f'(a_i)}\right)$

where $f(x)=(x-a_1)(x-a_2)\dots(x-a_n)$.

265. $\prod\limits_{n\geqslant i>k\geqslant 1}(x_i-x_k)$.

266. $2^{\frac{n(n-1)}{2}}\prod\limits_{n\geqslant i>k\geqslant 1}\cos\frac{\varphi_i+\varphi_k}{2}\prod\limits_{n\geqslant i>k\geqslant 1}\sin\frac{\varphi_i-\varphi_k}{2}$.

267. $\prod\limits_{n\geqslant i>k\geqslant 1}(x_i-x_k)$.

268. $2^{\frac{n(n-1)}{2}}a_{01}a_{02}\dots a_{0,\,n-1}\prod\limits_{n\geqslant i>k\geqslant 1}\sin\frac{\varphi_i+\varphi_k}{2}\prod\limits_{n\geqslant i>k\geqslant 1}\sin\frac{\varphi_k-\varphi_i}{2}$.

269. $\frac{1}{1!\,2!\dots(n-1)!}\prod\limits_{n\geqslant i>k\geqslant 1}(x_i-x_k)$.

271. $1!\,3!\,5!\dots(2n-1)!$ **272.** $\prod\limits_{i=1}^{n}\frac{x_i}{x_i-1}\prod\limits_{n\geqslant i>k\geqslant 1}(x_i-x_k)$.

273. $\prod\limits_{n+1\geqslant k>i\geqslant 1}(b_k a_i-a_k b_i)$. **274.** $\prod\limits_{1\leqslant i<k\leqslant n}\sin(\alpha_i-\alpha_k)$.

275. $\prod\limits_{1\leqslant i<k\leqslant n+1}(a_i-a_k)(a_i a_k-1)$.

276. $2^{(n-1)^2}\prod\limits_{n-1\geqslant i>k\geqslant 0}\sin\frac{\varphi_i+\varphi_k}{2}\prod\limits_{n-1\geqslant i>k\geqslant 0}\sin\frac{\varphi_k-\varphi_i}{2}$.

277. $2^{n(n+1)}\sin\alpha_0\sin\alpha_1$

$$\dots\sin\alpha_n\prod_{n\geqslant i>k\geqslant 0}\sin\frac{\alpha_i+\alpha_k}{2}\prod_{n\geqslant i>k\geqslant 0}\sin\frac{\alpha_i-\alpha_k}{2}.$$

278. $[x_1x_2\dots x_n-(x_1-1)\dots(x_n-1)]\prod\limits_{n\geqslant i>k\geqslant 1}(x_i-x_k)$.

279. $x_1x_2\dots x_n\left(\frac{1}{x_1}+\frac{1}{x_2}+\dots+\frac{1}{x_n}\right)\prod\limits_{n\geqslant i>k\geqslant 1}(x_i-x_k)$.

280. $(x_1+x_2+\dots+x_n)\prod\limits_{n\geqslant i>k\geqslant 1}(x_i-x_k)$.

281. $\sigma_{n-s}\prod\limits_{n\geqslant i>k\geqslant 1}(x_i-x_k)$ where σ_p forms a sum of all possible products of the numbers $x_1, x_2, \dots, x_n$ taken p at a time.

282. $[2x_1x_2\ldots x_n-(x_1-1)(x_2-1)\ldots(x_n-1)]\prod_{n\geqslant i>k\geqslant 1}(x_i-x_k).$

283. $x^2(x^2-1)^4$. **284.** $2x^3y(x-y)^6$.

285. $1!\,2!\,3!\ldots(n-1)!\,x^{\frac{n(n-1)}{2}}(y-x)^n.$

286. $1!\,2!\,3!\ldots(k-1)!\,x^{\frac{k(k-1)}{2}}(y_1-x)^k(y_2-x)^k$

$$\ldots(y_{n-k}-x)^k\prod_{n-k\geqslant i>j\geqslant 1}(y_i-y_j).$$

287. $(y-x)^{k(n-k)}$.

288. (b) 9, (c) 5, (e) 128, (f) $(a_1a_2-b_1b_2)(c_1c_2-d_1d_2)$,

(g) $(x_3-x_2)^2(x_3-x_1)^2(x_2-x_1)^2$,

(h) $(\lambda^2-a^2)(\alpha-\beta)^{n-1}[\alpha+(n-1)\beta]$,

(k) $(x_4-x_3)[(x_3-x_2)(x_4-x_2)-2(x_3-x_1)(x_4-x_1)]$,

(m) $27(a+2)^3(a-1)^6[3(a+2)^2-4x^2][3(a-1)^2-4x^2]^3$.

Remark. This problem is a particular case of Problem 537.

289. (a) $\begin{vmatrix} -5 & -2 \\ -8 & 4 \end{vmatrix}$, (b) $\begin{vmatrix} 2 & 8 & 17 \\ 11 & -6 & 5 \\ 3 & 8 & -3 \end{vmatrix}$,

(c) $\begin{vmatrix} 7 & 5 & -3 & 3 \\ -1 & 5 & -3 & 3 \\ -4 & -4 & -5 & 4 \\ -4 & -4 & 0 & 3 \end{vmatrix}$.

290. (a) 24, (b) 18,
(c) $(a+b+c+d)(a+b-c-d)(a-b+c-d)(a-b-c+d)$.

291. (a) 256, (b) 78400, (c) $(a^2+b^2+c^2+d^2)^4$.

292. $D\prod_{n\geqslant i>k\geqslant 1}(x_i-x_k)$.

293. (a) $C_n^1C_n^2\ldots C_n^n\prod_{n\geqslant i>k\geqslant 0}(a_i-a_k)(b_k-b_i)$,

(b) $\prod_{n\geqslant i>k\geqslant 1}(\alpha_i-\alpha_k)(\beta_i-\beta_k)$.

294. 0 if $n\geqslant 2$. **295.** $\prod_{i=1}^{n}(x-x_i)\prod_{n\geqslant i>k\geqslant 1}(x_i-x_k)^2$.

296. $-(a^2+b^2+c^2+d^2+l^2+m^2+n^2+p^2)^4$.

297. $4\sin^4\varphi$. **298.** $4\sin^4\varphi$.

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299. Denote the desired determinant by Δ. Squaring yields $|\Delta|=n^{\frac{n}{2}}$ On the other hand, $\Delta=\prod\limits_{n-1\geqslant k>s\geqslant 0}(\varepsilon^k-\varepsilon^s)$.

We assume $\varepsilon_1=\cos\frac{\pi}{n}+i\sin\frac{\pi}{n}$. Then $\varepsilon=\varepsilon_1^2$ and

$$\Delta=\prod_{n-1\geqslant k>s\geqslant 0}(\varepsilon^k-\varepsilon^s)=\prod\varepsilon_1^{k+s}\prod(\varepsilon_1^{k-s}-\varepsilon_1^{-k+s})$$

$$=\prod\varepsilon_1^{k+s}\cdot i^{\frac{n(n-1)}{2}}\prod 2\sin\frac{(k-s)\pi}{n}.$$

Furthermore, $\sin\frac{(k-s)\pi}{n}>0$ for all k, s. Hence

$$n^{\frac{n}{2}}=|\Delta|=\left|\prod 2\sin\frac{(k-s)\pi}{n}\right|=\prod 2\sin\frac{(k-s)\pi}{n}.$$

Therefore

$$\Delta=n^{\frac{n}{2}}i^{\frac{n(n-1)}{2}}\prod_{n-1\geqslant k>s\geqslant 0}\varepsilon_1^{k+s}=n^{\frac{n}{2}}i^{\frac{n(n-1)}{2}}\varepsilon_1^{\frac{n(n-1)^2}{2}}$$

$$=n^{\frac{n}{2}}i^{\frac{n(n-1)}{2}+(n-1)^2}=n^{\frac{n}{2}}i^{-\frac{(n-1)(n+2)}{2}}.$$

300. $\prod\limits_{k=0}^{n-1}(a_0+a_1\varepsilon_k+a_2\varepsilon_k^2+\ldots+a_{n-1}\varepsilon_k^{n-1})$ where $\varepsilon_k=\cos\frac{2k\pi}{n}+i\sin\frac{2k\pi}{n}$.

301. $x^4-y^4+z^4-u^4+4xy^2z+4xzu^2-4x^2yu-4yz^2u-2x^2z^2+2y^2u^2$.

303. 2^{n-1} if n is odd, 0 if n is even.

304. $(-1)^n\dfrac{[(n+1)a^n-1]^n-n^na^{n(n+1)}}{(1-a^n)^2}$.

305. $(-1)^{n-1}(n-1)\prod\limits_{k=0}^{n-1}(a_1+a_2\varepsilon_k+\ldots+a_n\varepsilon_k^{n-1})$ where

$\varepsilon_k=\cos\frac{2k\pi}{n}+i\sin\frac{2k\pi}{n}$.

306. $\varphi_0(t)\varphi_1(t)\ldots\varphi_{n-1}(t)$ where $\varphi_k(t)=\dfrac{(t+\varepsilon_k)^n-1}{t}$;

$\varepsilon_k=\cos\frac{2k\pi}{n}+i\sin\frac{2k\pi}{n}$.

According to the result of Problem 102, the answer can be represented as $\prod_{k=1}^{n-1} [t^n-(\varepsilon_k-1)^n]$.

307. $(-2)^{n-1}(n-2p)$ if $(n, p)=1$, 0 if $(n, p)\neq 1$.

308. $2^{n-2}\left(\cos^n \frac{\pi}{n}-1\right)$.

309. $2^{n-2}\sin^{n-2}\frac{n\Theta}{2}\left[\sin^n\frac{(n+2)\Theta}{2}-\sin^n\frac{n\Theta}{2}\right]$.

310. $(-1)^n 2^{n-2}\sin^{n-2}\frac{nh}{2}\left[\cos^n\left(a+\frac{nh}{2}\right)-\cos^n\left(a+\frac{(n-2)h}{2}\right)\right]$.

311. $(-1)^{n-1}\frac{(n+1)(2n+1)}{12}n^{n-2}[(n+2)^n-n^n]$.

313. $\prod_{k=0}^{n-1}(a_1+a_2\varepsilon_k+a_3\varepsilon_k^2+\ldots+a_n\varepsilon_k^{n-1})$ where $\varepsilon_k=\cos\frac{(2k+1)\pi}{n}+i\sin\frac{(2k+1)\pi}{n}$.

315. $\prod_{i=1}^{n}(a_1+a_2\rho_i+a_3\rho_i^2+\ldots+a_n\rho_i^{n-1})$ where $\rho_1, \rho_2, \ldots, \rho_n$ are nth roots of μ.

318. The solution of Problem 223. Adding 1 to all elements of the determinant $\begin{vmatrix} a_1 & 0 & \ldots & 0 \\ 0 & a_2 & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \ldots & a_n \end{vmatrix}$, we get the determinant Δ.

We have $\Delta=a_1a_2\ldots a_n+\sum_{k=1}^{n}\sum_{i=1}^{n}A_{ik}$,

$$\sum_{k=1}^{n}\sum_{i=1}^{n}A_{ik}=a_1a_2\ldots a_n\left(\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_n}\right).$$

The solution of Problem 250. Denote by Δ the determinant to be evaluated. We have

$$\Delta=(a_1-x)(a_2-x)\ldots(a_n-x)+x\sum A_{ik},$$

$$\Delta=(a_1-y)(a_2-y)\ldots(a_n-y)+y\sum A_{ik}$$

where $\sum A_{ik}$ is the sum of the cofactors of all elements of Δ.

Δ is readily determined from the system of equations.

323. $$\prod_{1\leqslant i<k\leqslant n}(a_i-a_k)\prod_{1\leqslant i<k\leqslant n}(b_i-b_k)\frac{1}{\prod_{i=1}^{n}f(a_i)}$$

where $f(x)=(x+b_1)\dots(x+b_n)$.

325. $$\frac{[c+\sqrt{c^2-4ab}]^{n+1}-[c-\sqrt{c^2-4ab}]^{n+1}}{2^{n+1}\sqrt{c^2-4ab}}.$$

326. $$\frac{[p+\sqrt{p^2-4q}]^n+[p-\sqrt{p^2-4q}]^n}{2^n}.$$

327. $x^n+a_1x^{n-1}+a_2x^{n-2}+\dots+a_n$, where a_k is the sum of all kth-order minors of the determinant

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

obtained from it by deleting $n-k$ rows with indices $\alpha_1, \alpha_2, \dots, \alpha_{n-k}$ and columns with the same indices.

328. $(n+1)^{n-1}$. **329.** $(x-n)^{n+1}$.

330. $(x^2-1^2)(x^2-3^2)\dots[x^2-(2m-1)^2]$ if $n=2m$,

$x(x^2-2^2)(x^2-4^2)\dots(x^2-4m^2)$ if $n=2m+1$.

331. $(x+na-n)[x+(n-2)a-n+1][x+(n-4)a-n+2]\dots(x-na)$.

332. $(-1)^{\frac{n(n-1)}{2}}[(n-1)!]^n$. **333.** $\dfrac{[1!\,2!\dots(n-1)!]^3}{n!\,(n+1)!\dots(2n-1)!}$.

334. $\dfrac{\Delta(a_1, a_2, \dots, a_n)\,\Delta(b_1, b_2, \dots, b_n)}{(1, 2, \dots, n)}$ where Δ is the Vandermonde determinant.

CHAPTER 3

SYSTEMS OF LINEAR EQUATIONS

335. $x_1=3$, $x_2=x_3=1$.
336. $x_1=1$, $x_2=2$, $x_3=-2$.
337. $x_1=2$, $x_2=-2$, $x_3=3$.
338. $x_1=3$, $x_2=4$, $x_3=5$.
339. $x_1=x_2=-1$, $x_3=0$, $x_4=1$.
340. $x_1=1$, $x_2=2$, $x_3=-1$, $x_4=-2$.
341. $x_1=-2$, $x_2=2$, $x_3=-3$, $x_4=3$.
342. $x_1=1$, $x_2=2$, $x_3=1$, $x_4=-1$.
343. $x_1=2$, $x_2=x_3=x_4=0$.
344. $x_1=x_2=x_3=x_4=0$.
345. $x_1=1$, $x_2=-1$, $x_3=0$, $x_4=2$,

346. $x_1=x_2=x_3=x_4=x_5=0$.

347. $x_1=x_2=x_3=x_4=0$.

348. $x_1=1,\ x_2=-1,\ x_3=1,\ x_4=-1,\ x_5=1$.

349. $x_1=x_2=x_3=x_4=x_5=0$.

350. $x_1=1,\ x_2=-1,\ x_3=1,\ x_4=-1,\ x_5=1$.

351. $x_1=0,\ x_2=2,\ x_3=-2,\ x_4=0,\ x_5=3$.

352. $x_1=2,\ x_2=0,\ x_3=-2,\ x_4=-2,\ x_5=1$.

353. The determinant of the system is equal to zero, since the system has a nonzero solution.

354. The determinant of the system is equal to $-(a^2+b^2+c^2+d^2)^2$.

355. $$x_i=\frac{a\sum_{k=1}^{n} a_k-a_i[(n-1)\alpha+\beta]}{(\alpha-\beta)[(n-1)\alpha+\beta]}.$$

356. $$x_i=-\frac{f(\beta_i)}{\varphi'(\beta_i)}\quad\text{where}\quad \begin{aligned}f(x)&=(x-b_1)(x-b_2)\ldots(x-b_n),\\ \varphi(x)&=(x-\beta_1)(x-\beta_2)\ldots(x-\beta_n).\end{aligned}$$

357. $$x_i=\frac{f(t)}{(t-\alpha_i)f'(\alpha_i)}\quad\text{where}\quad f(x)=(x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_n).$$

358. $$x_s=\sum_{i=1}^{n}\frac{(-1)^{n+s}u_i}{f'(\alpha_i)}\varphi_{s,i}\quad\text{where}\quad f(x)=(x-\alpha_1)\ldots(x-\alpha_n);$$

$\varphi_{s,i}=\sum \alpha_{t_1}\alpha_{t_2}\ldots\alpha_{t_{n-s}}$; the sum is taken over all combinations $t_1, t_2, \ldots, t_{n-s}$ of $1, 2, \ldots, i-1, i+1, \ldots, n$.

359. $$x_s=\sum_{i=1}^{n}\frac{(-1)^{n+i}u_i}{f'(\alpha_s)}\varphi_{i,s}\quad\text{where}\quad f(x)=(x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_n);$$

$$\varphi_{i,s}=\sum \alpha_{t_1}\alpha_{t_2}\ldots\alpha_{t_{n-i}};$$

here, the sum is taken over all combinations $t_1, t_2, \ldots, t_{n-i}$ of $1, 2, \ldots, s-1, s+1, \ldots, n$.

360. $$x_i=\frac{(-1)^n a_{n-i}}{n!}\quad\text{where}\quad x^n+a_1x^{n-1}+\ldots+a_n=(x-1)(x-2)\ldots(x-n).$$

361. $C_m^k\cdot C_n^k$.

365. (a) It does not change or it increases by unity; (b) it does not change or it increases by unity or by two.

366. 2. **367.** 3. **368.** 2. **369.** 2. **370.** 3. **371.** 3. **372.** 4. **373.** 3. **374.** 2. **375.** 3. **376.** 5. **377.** 6. **378.** 5. **379.** 3. **380.** 4.

383. The forms are independent. **384.** $2y_1-y_2-y_3=0$.

385. $y_1+3y_2-y_3=0,\ 2y_1-y_2-y_4=0$.

386. The forms are independent.

387. $y_1+y_2-y_3-y_4=0$.
388. $y_1-y_2+y_3=0$, $5y_1-4y_2+y_4=0$.
389. The forms are independent.
390. The forms are independent.
391. $y_1+y_2-y_3-y_4=0$. **392.** $2y_1-y_2-y_3=0$.
393. $3y_1-y_2-y_3=0$, $y_1-y_2-y_4=0$.
394. The forms are independent. **395.** $y_1-y_2-y_3-y_4=0$.
396. $3y_1-2y_2-y_3+y_4=0$, $y_1-y_2+2y_3-y_5=0$.
397. $\lambda=10$, $3y_1+2y_2-5y_3-y_4=0$.
398. $x_3=2x_2-x_1$, $x_4=1$. **399.** $\lambda=5$.
400. The system has no solutions. **401.** $x_1=1$, $x_2=2$, $x_3=-2$.
402. $x_1=1$, $x_2=2$, $x_3=1$. **403.** $x_1=-\frac{11x_3}{7}$, $x_2=-\frac{x_3}{7}$.
404. The system has no solutions.
405. $x_1=0$, $x_2=2$, $x_3=\frac{5}{3}$, $x_4=-\frac{4}{3}$.
406. $x_1=-8$, $x_2=3+x_4$, $x_3=6+2x_4$.
407. $x_1=2$, $x_2=x_3=x_4=1$. **408.** $x_1=x_2=x_3=x_4=0$.
409. $x_1=\frac{3x_3-13x_4}{17}$, $x_2=\frac{19x_3-20x_4}{17}$.
410. $x_1=\frac{7}{6}x_5-x_3$, $x_2=\frac{5}{6}x_5+x_3$, $x_4=\frac{x_5}{3}$.
411. $x_1=-16+x_3+x_4+5x_5$, $x_2=23-2x_3-2x_4-6x_5$.
412. $x_1=\frac{-4x_4+7x_5}{8}$, $x_2=\frac{-4x_4+5x_5}{8}$, $x_3=\frac{4x_4-5x_5}{8}$.
413. $x_1=x_2=x_3=0$, $x_4=x_5$.
414. $x_1=\frac{1+x_5}{3}$, $x_2=\frac{1+3x_3+3x_4-5x_5}{3}$.
415. $x_1=\frac{2+x_5}{3}$, $x_2=\frac{1+3x_3-3x_4-5x_5}{6}$.
416. The system has no solutions.
417. The system has no solutions.
418. $x_1=-\frac{x_5}{2}$, $x_2=-1-\frac{x_5}{2}$, $x_3=0$, $x_4=-1-\frac{x_5}{2}$.
419. $x_1=\frac{1+5x_4}{6}$, $x_2=\frac{1-7x_4}{6}$, $x_3=\frac{1+5x_4}{6}$.
420. The system has no solutions.
421. $x=\frac{b^2+c^2-a^2}{2bc}$, $y=\frac{a^2+c^2-b^2}{2ac}$, $z=\frac{a^2+b^2-c^2}{2ab}$.
422. If $(\lambda-1)(\lambda+2)\neq 0$, $x=-\frac{\lambda+1}{\lambda+2}$, $y=\frac{1}{\lambda+2}$, $z=\frac{(\lambda+1)^2}{\lambda+2}$.
If $\lambda=1$, the system has solutions dependent on two parameters. If $\lambda=-2$, the system has no solutions.

423. If $(\lambda-1)(\lambda+3)\neq 0$, $x=-\dfrac{\lambda^2+2\lambda+2}{\lambda+3}$,

$$y=-\frac{\lambda^2+\lambda-1}{\lambda+3},\quad z=\frac{2\lambda+1}{\lambda+3},\quad t=-\frac{\lambda^3+3\lambda^2+2\lambda+1}{\lambda+3}.$$

If $\lambda=1$, the system has solutions dependent on three parameters.
If $\lambda=-3$, the system has no solutions.

424. If a, b, c are all distinct,

$x=abc$, $y=-(ab+ac+bc)$, $z=a+b+c$.

If among a, b, c two are equal, the solutions depend on one parameter.
If $a=b=c$, the solutions depend on two parameters.

425. If a, b, c are all distinct,

$$x=\frac{(b-d)(c-d)}{(b-a)(c-a)},\quad y=\frac{(a-d)(c-d)}{(a-b)(c-b)},\quad z=\frac{(a-d)(b-d)}{(a-c)(b-c)}.$$

If $a=b$, $a\neq c$, $d=a$ or $d=c$, the solutions depend on one parameter.
If $b=c$, $a\neq b$, $d=a$ or $d=b$, the solutions depend on one parameter.
If $a=c$, $a\neq b$, $d=a$ or $d=b$, the solutions depend on one parameter.
If $a=b=c=d$, the solutions depend on two parameters.
In all other cases, the system has no solutions.

426. If $b(a-1)\neq 0$, $x=\dfrac{2b-1}{b(a-1)}$, $y=\dfrac{1}{b}$, $z=\dfrac{2ab-4b+1}{b(a-1)}$.

If $a=1$, $b=\dfrac{1}{2}$, the solutions depend on one parameter.
In all other cases, the system has no solutions.

427. If $b(a-1)(a+2)\neq 0$, $x=z=\dfrac{a-b}{(a-1)(a+2)}$, $y=\dfrac{ab+b-2}{b(a-1)(a+2)}$.

If $a=-2$, $b=-2$, the solutions depend on one parameter.
If $a=1$, $b=1$, the solutions depend on two parameters.
In all other cases the system has no solutions.

428. If $(\alpha-1)(\alpha+2)\neq 0$, $x=\dfrac{m\alpha+m-n-p}{(\alpha+2)(\alpha-1)}$,

$$y=\frac{n\alpha+n-m-p}{(\alpha+2)(\alpha-1)},\quad z=\frac{p\alpha+p-m-n}{(\alpha+2)(\alpha-1)}.$$

If $\alpha=-2$ and $m+n+p=0$, the solutions depend on one parameter.
If $\alpha=1$ and $m=n=p$, the solutions depend on two parameters.
In all other cases the system has no solutions.

429. If $a(a-b)\neq 0$, $x=\dfrac{a^2(b-1)}{b-a}$, $y=\dfrac{b(a^2-1)}{a(a-b)}$, $z=\dfrac{a-1}{a(b-a)}$.

If $a=b=1$, the solutions depend on two parameters.
In all other cases the system has no solutions.

430. $\Delta=\lambda^2(\lambda-1)$. For $\lambda=0$, $\lambda=1$, the system is inconsistent.

431. $\Delta=-2\lambda$. If $\lambda\neq 0$, $x=1-\lambda$, $y=\lambda$, $z=0$. If $\lambda=0$, $x=1$, $z=0$, y is arbitrary.

432. $\Delta=(k-1)^2(k+1)$. If $k=1$, the solution depends on one parameter. If $k=-1$, the system is inconsistent.

433. $\Delta=a(b-1)(b+1)$.

If $a=0$, $b=5$, $y=-\frac{1}{3}$, $z=\frac{4}{3}$, x is arbitrary.

If $a=0$, $b\neq 1$ and $b\neq 5$, the system is inconsistent.

If $b=1$, $z=0$, $y=1-ax$, x is arbitrary.

If $b=-1$, the system is inconsistent.

434. (a) $\Delta=-m(m+2)$. For $m=0$ and $m=-2$ the system is inconsistent.

(b) $\Delta=m(m^2-1)$. If $m=0$, $m=1$, the system is inconsistent. If $m=-1$, the solution depends on one parameter.

(c) $\Delta=\lambda(\lambda-1)(\lambda+1)$. If $\lambda=1$, $\lambda=-1$, the system is inconsistent. If $\lambda=0$ the solution depends on one parameter.

435. (a) $\Delta=3(c+1)(c-1)^2$. If $c=-1$, the system is inconsistent.

If $c=1$, the solution depends on two parameters.

(b) $\Delta=(\lambda-1)(\lambda-2)(\lambda-3)$. If $\lambda=2$, $\lambda=3$, the system is inconsistent. If $\lambda=1$, the solution depends on one parameter.

(c) $\Delta=d(d-1)(d+2)$. If $d=1$, $d=-2$, the system is inconsistent. If $d=0$, the solution depends on one parameter.

(d) $\Delta=(a-1)^2(a+1)$. If $a=-1$, the system is inconsistent. If $a=1$, the solution depends on two parameters.

436. $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}=0.$

437. If and only if $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}=0.$

438. If and only if $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}=0.$

439. If and only if $\begin{vmatrix} x_0^2+y_0^2 & x_0 & y_0 & 1 \\ x_1^2+y_1^2 & x_1 & y_1 & 1 \\ x_2^2+y_2^2 & x_2 & y_3 & 1 \\ x_3^2+y_3^2 & x_3 & y_3 & 1 \end{vmatrix}=0.$

440. $(x-1)^2+(y-1)^2=1$. **441.** $y^2-y=0$.

442. $y=x^3-1$.

443. $\begin{vmatrix} y & x^n & x^{n-1} & \dots & x^2 & x & 1 \\ y_0 & x_0^n & x_0^{n-1} & \dots & x_0^2 & x_0 & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ y_n & x_n^n & x_n^{n-1} & \dots & x_n^2 & x_n & 1 \end{vmatrix}=0.$

444. If and only if $\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}=0.$

445. $x^2+y^2+z^2-x-y-z=0$.

446. If and only if the rank of the matrix

$$\begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ \cdots & \cdots & \cdots \\ x_n & y_n & 1 \end{pmatrix}$$

is less than three.

447. If and only if the rank of the matrix

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ \cdots & \cdots & \cdots \\ a_n & b_n & c_n \end{pmatrix}$$

is less than three.

448. In one plane if and only if the rank of the matrix

$$\begin{pmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ \cdots & \cdots & \cdots & \cdots \\ x_n & y_n & z_n & 1 \end{pmatrix}$$

is less than four; on one straight line if and only if the rank of the matrix is less than three.

449. All planes pass through one point only when the rank of the matrix

$$\begin{pmatrix} A_1 & B_1 & C_1 & D_1 \\ A_2 & B_2 & C_2 & D_2 \\ \cdots & \cdots & \cdots & \cdots \\ A_n & B_n & C_n & D_n \end{pmatrix}$$

is less than four; through one straight line only when the rank of this matrix is less than three.

450. $$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1,\,n-1} & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2,\,n-1} & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \dots & a_{n,\,n-1} & a_{nn} \end{vmatrix} = 0.$$

453. No. **454.** For example, $\begin{pmatrix} 1 & -2 & 1 & 0 & 0 \\ 1 & -2 & 0 & 1 & 0 \\ 5 & -6 & 0 & 0 & 1 \end{pmatrix}$.

455. Yes.

456. *Solution.* Let

$$A = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_{r1} & \alpha_{r2} & \dots & \alpha_{rn} \end{pmatrix}, \quad B = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \dots & \lambda_{1r} \\ \lambda_{21} & \lambda_{22} & \dots & \lambda_{2r} \\ \cdots & \cdots & \cdots & \cdots \\ \lambda_{r1} & \lambda_{r2} & \dots & \lambda_{rr} \end{pmatrix},$$

$$BA=\begin{pmatrix} \sum_{s=1}^{r} \lambda_{1s} \alpha_{s1} & \dots & \sum_{s=1}^{r} \lambda_{1s} \alpha_{sn} \\ \dots & \dots & \dots \\ \sum_{s=1}^{r} \lambda_{rs} \alpha_{s1} & \dots & \sum_{s=1}^{r} \lambda_{rs} \alpha_{sn} \end{pmatrix}.$$

It is immediately obvious that the rows of the matrix BA are solutions of the system. Besides, since $|B|\neq 0$, $A=B^{-1}(BA)$, i. e., the solutions written down by the matrix A are linear combinations of the solutions written down by the matrix BA.

457. *Solution.* Let

$$A=\begin{pmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2n} \\ \dots & \dots & \dots & \dots \\ \alpha_{r1} & \alpha_{r2} & \dots & \alpha_{rn} \end{pmatrix}, \quad C=\begin{pmatrix} \gamma_{11} & \gamma_{12} & \dots & \gamma_{1n} \\ \gamma_{21} & \gamma_{22} & \dots & \gamma_{2n} \\ \dots & \dots & \dots & \dots \\ \gamma_{r1} & \gamma_{r2} & \dots & \gamma_{rn} \end{pmatrix}.$$

Since C is a fundamental system of solutions, $\alpha_{11}=\lambda_{11}\gamma_{11}+\lambda_{12}\gamma_{21}+\dots+\lambda_{1r}\gamma_r$ and so on, that is, $A=BC$, where

$$B=\begin{pmatrix} \lambda_{11} & \lambda_{12} & \dots & \lambda_{1r} \\ \dots & \dots & \dots & \dots \\ \lambda_{r1} & \lambda_{r2} & \dots & \lambda_{rr} \end{pmatrix}.$$

On the other hand, A is also a fundamental system of solutions, and therefore $|B|\neq 0$.

459. For example,

$x_1=c_1+c_2+5c_3$, $x_2=-2c_1-2c_2-6c_3$, $x_3=c_1$, $x_4=c_2$, $x_5=c_3$

(see the answer to Problem 454).

460. $x_1=11c$, $x_2=c$, $x_3=-7c$.

461. No. 408. $x_1=x_2=x_3=x_4=0$.

No. 409. $x_1=3c_1+13c_2$, $x_2=19c_1+20c_2$, $x_3=17c_1$, $x_4=-17c_2$.

No. 410. $x_1=c_1+7c_2$, $x_2=-c_1+5c_2$, $x_3=-c_1$, $x_4=2c_2$, $x_5=6c_2$.

No. 412. $x_1=c_1+7c_2$, $x_2=c_1+5c_2$, $x_3=-c_1-5c_2$, $x_4=-2c_1$; $x_5=8c_2$.

No. 413. $x_1=0$, $x_2=0$, $x_3=0$, $x_4=c$, $x_5=c$.

462. $x_1=-16+c_1+c_2+5c_3$,

$x_2=23-2c_1-2c_2-6c_3$,

$x_3=c_1$; $x_4=c_2$; $x_5=c_3$.

463. No. 406. $x_1=-8$, $x_2=3+c$, $x_3=6+2c$, $x_4=c$.

No. 414. $x_1=c_3$, $x_2=2+c_1+c_2-5c_3$, $x_2=c_1$, $x_4=c_2$, $x_5=-1+3c_3$.

No. 415. $x_1=1+2c_3$, $x_2=1+c_1-c_2+5c_3$, $x_3=2c_1$, $x_4=2c_2$, $x_5=1+6c_3$.

CHAPTER 4

MATRICES

464. (a) $\begin{pmatrix} 3 & -1 \\ 5 & -1 \end{pmatrix}$, (b) $\begin{pmatrix} -9 & 13 \\ 15 & 4 \end{pmatrix}$,

(c) $\begin{pmatrix} 6 & 2 & -1 \\ 6 & 1 & 1 \\ 8 & -1 & 4 \end{pmatrix}$, (d) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, (e) $\begin{pmatrix} 1 & 9 & 15 \\ -5 & 5 & 9 \\ 12 & 26 & 32 \end{pmatrix}$,

(f) $\begin{pmatrix} a+b+c & a^2+b^2+c^2 & b^2+2ac \\ a+b+c & b^2+2ac & a^2+b^2+c^2 \\ 3 & a+b+c & a+b+c \end{pmatrix}$.

465. (a) $\begin{pmatrix} 7 & 4 & 4 \\ 9 & 4 & 3 \\ 3 & 3 & 4 \end{pmatrix}$, (b) $\begin{pmatrix} 15 & 20 \\ 20 & 35 \end{pmatrix}$, (c) $\begin{pmatrix} 3 & -2 \\ 4 & 8 \end{pmatrix}$,

(d) $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$, (e) $\begin{pmatrix} \cos n\varphi & -\sin n\varphi \\ \sin n\varphi & \cos n\varphi \end{pmatrix}$.

466. $\begin{pmatrix} 1 & \frac{\alpha}{n} \\ -\frac{\alpha}{n} & 1 \end{pmatrix} = \sqrt{1+\frac{\alpha^2}{n^2}} \cdot \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}$

where $\tan\varphi = \frac{\alpha}{n}$. Hence,

$$\begin{pmatrix} 1 & \frac{\alpha}{n} \\ -\frac{\alpha}{n} & 1 \end{pmatrix}^n = \left(1+\frac{\alpha^2}{n^2}\right)^{\frac{n}{2}} \cdot \begin{pmatrix} \cos n\varphi & \sin n\varphi \\ -\sin n\varphi & \cos n\varphi \end{pmatrix}.$$

The limit of the first factor is equal to 1. $\lim\limits_{n\to\infty} n\varphi = \alpha \lim\limits_{\varphi\to 0} \frac{\varphi}{\tan\varphi} = \alpha$. Therefore

$$\lim \begin{pmatrix} 1 & \frac{\alpha}{n} \\ -\frac{\alpha}{n} & 1 \end{pmatrix}^n = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}.$$

467. (a) $(A+B)^2 = A^2+AB+BA+B^2 = A^2+2AB+B^2$.

(b) $(A+B)(A-B) = A^2-AB+BA-B^2 = A^2-B^2$.

(c) The proof is by induction.

468. (a) $\begin{pmatrix} -10 & -4 & -7 \\ 6 & 14 & 4 \\ -7 & 5 & -4 \end{pmatrix}$, (b) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

469. (a) $\begin{pmatrix} x & 2y \\ -y & x-2y \end{pmatrix} = (x-y)\,E + y\,A$,

(b) $\begin{pmatrix} x & y \\ 0 & x \end{pmatrix} = (x-y)\,E + yA$,

(c) $\begin{pmatrix} x & y & 0 \\ u & v & 0 \\ 3t- \quad 3x-u & t-3y-v & t \end{pmatrix}$.

470. (a) $\begin{pmatrix} 5 & 1 & 3 \\ 8 & 0 & 3 \\ -2 & 1 & -2 \end{pmatrix}$, (b) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

471. It is verified by direct computation.

472. The polynomials $F(x) = a_0 + a_1 x + \ldots + a_m x^m$, such that $F(A) = 0$, exist because the equality $F(A) = a_0 E + a_1 A + \ldots + a_m A^m = 0$ is equivalent to a system of n^2 homogeneous linear equations in $m+1$ unknowns $a_0, a_1, \ldots, a_m$, which system probably has nontrivial solutions for $m \geqslant n^2$. Let $F(x)$ be some polynomial for which $F(A) = 0$ and let $f(x)$ be a polynomial of lowest degree among the polynomials having this property. Then $F(x) = f(x)\,q(x) + r(x)$, where $r(x)$ is a polynomial of degree less than that of $f(x)$. We have $r(A) = F(A) - f(A)\,q(A) = 0$, hence, $r(x) = 0$, otherwise there would be a contradiction with the choice of $f(x)$. Thus, $F(x) = f(x)\,q(x)$.

473. Let

$$A = \begin{pmatrix} a_{11} & a_{12} \cdots a_{1n} \\ \cdots & \cdots \\ a_{n1} & a_{n2} \cdots a_{nn} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} \cdots b_{1n} \\ \cdots \\ b_{n1} \cdots b_{nn} \end{pmatrix}.$$

Then the sum of the diagonal elements of the matrix AB is equal to $\sum_{i=1}^{n} \sum_{k=1}^{n} a_{ik} b_{ki}$. The sum of the diagonal elements of the matrix BA is exactly the same. Hence, the sum of the diagonal elements of the matrix $AB - BA$ is equal to zero, and the equality $AB - BA = E$ is impossible.

Remark. The result is not valid for matrices with elements of a field of characteristic $p \neq 0$. Indeed, in a field of characteristic p, for matrices of order p,

$$A = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \ldots & 1 \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 & \ldots & & 0 \\ 1 & 0 & 0 & \ldots & & 0 \\ 0 & 2 & 0 & \ldots & & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \ldots & p-1 & 0 \end{pmatrix}$$

we have $AB - BA = E$.

474. $(E - A)(E + A + A^2 + \ldots + A^{k-1}) = E - A^k = E$.

475. $\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$, $\quad bc = -a^2$.

476. If $A^3=0$, then $A^2=0$ also. Indeed, if $A^3=0$, then $|A|=0$. Hence (see 471), $A^2=(a+d)A$, $0=A^3=(a+d)A^2=(a+d)^2A$, whence $a+d=0$ and $A^2=0$.

477. $\pm E$, $\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$, $\quad a^2=1-bc$.

478. If $A=0$, then X is any matrix. If $|A|\neq 0$, then $X=0$. Finally, if $|A|=0$, but $A\neq 0$, then the rows of matrix A are proportional. Let $\alpha:\beta$ be the ratio of corresponding elements of the first and second rows of the matrix A. Then $X=\begin{pmatrix} -\beta x & \alpha x \\ -\beta y & \alpha y \end{pmatrix}$ for arbitrary x, y.

479. Let $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

(1) If $A\neq 0$, but $a+d=0$, $ad-bc=0$, then there are no solutions;

(2) if $a+d\neq 0$, $(a-d)^2+4bc=0$, $(a-d)$, b, c are not equal to zero simultaneously, then there are two solutions:

$$X=\pm\frac{1}{2\sqrt{2(a+d)}}\begin{pmatrix} 3a+d & 2b \\ 2c & a+3d \end{pmatrix};$$

(3) if $a+d\neq 0$, $ad-bc=0$, then there are two solutions:

$$X=\pm\frac{1}{\sqrt{a+d}}\begin{pmatrix} a & b \\ c & d \end{pmatrix};$$

(4) if $ad-bc\neq 0$, $(a-d)^2+4bc\neq 0$, then there are four solutions:

$$X=\frac{1}{\lambda}\begin{pmatrix} \dfrac{\lambda^2+a-d}{2} & b \\ c & \dfrac{\lambda^2-a+d}{2} \end{pmatrix}$$

where $\lambda=\pm\sqrt{a+d\pm 2\sqrt{ad-bc}}$;

(5) if $a-d=b=c=0$, then there are infinitely many solutions: $X=\pm\sqrt{a}E$ and $X=\begin{pmatrix} x & y \\ z & -x \end{pmatrix}$, where x, y, z are connected by the relation $x^2+yz=a$.

480. (a) $\begin{pmatrix} 5 & -2 \\ -2 & 1 \end{pmatrix}$, (b) $\dfrac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$,

(c) $\begin{pmatrix} 1 & -2 & 7 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$, (d) $\begin{pmatrix} 1 & -3 & 11 & -38 \\ 0 & 1 & -2 & 7 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$,

(e) $\begin{pmatrix} 1 & -4 & -3 \\ 1 & -5 & -3 \\ -1 & 6 & 4 \end{pmatrix}$, (f) $\dfrac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$,

(g) $$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -3 & 2 & 0 & 0 \\ 31 & -19 & 3 & -4 \\ -23 & 14 & -2 & 3 \end{pmatrix},$$

(h) $$\frac{1}{n-1}\begin{pmatrix} 2-n & 1 & 1 & & 1 \\ 1 & 2-n & 1 & & 1 \\ 1 & 1 & 2-n & \ldots & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & 1 & \ldots & 2-n \end{pmatrix},$$

(i) $$\frac{1}{n}\begin{pmatrix} 1 & 1 & 1 & & 1 \\ 1 & \varepsilon^{-1} & \varepsilon^{-2} & \ldots & \varepsilon^{-n+1} \\ 1 & \varepsilon^{-2} & \varepsilon^{-4} & \ldots & \varepsilon^{-2n+2} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \varepsilon^{-n+1} & \varepsilon^{-2n+2} & \ldots & \varepsilon^{-(n-1)^2} \end{pmatrix},$$

(j) $$\frac{1}{n+1}\begin{pmatrix} 1\cdot n & 1\cdot(n-1) & 1\cdot(n-2) & \ldots & 1\cdot 1 \\ 1\cdot(n-1) & 2\cdot(n-1) & 2\cdot(n-2) & \ldots & 2\cdot 1 \\ 1\cdot(n-2) & 2\cdot(n-2) & 3\cdot(n-2) & \ldots & 3\cdot 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1\cdot 1 & 2\cdot 1 & 3\cdot 1 & \ldots & n\cdot 1 \end{pmatrix},$$

(k) $$\frac{1}{2n^3}\begin{pmatrix} 2-n^2 & 2+n^2 & 2 & \ldots & 2 \\ 2 & 2-n^2 & 2+n^2 & \ldots & 2 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 2+n^2 & 2 & 2 & \ldots & 2-n^2 \end{pmatrix},$$

(l) $$\frac{1}{d}\begin{pmatrix} b_1c_1+d & b_2c_1 & \ldots & b_nc_1 & -c_1 \\ b_1c_2 & b_2c_2+d & \ldots & b_nc_2 & -c_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ b_1c_n & b_2c_n & \ldots & b_nc_n+d & -c_n \\ -b_1 & -b_2 & \ldots & -b_n & 1 \end{pmatrix}$$

where $d=a-b_1c_1-b_2c_2-\ldots-b_nc_n$,

(m) $$\frac{1}{f}\begin{pmatrix} f-f_0x^n & xf-f_1x^n & \ldots & x^{n-1}f-f_{n-1}x^n & x^n \\ -f_0x^{n-1} & f-f_1x^{n-1} & \ldots & x^{n-2}f-f_{n-1}x^{n-1} & x^{n-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ -f_0x & -f_1x & \ldots & f-f_{n-1}x & x \\ -f_0 & -f_1 & \ldots & -f_{n-1} & 1 \end{pmatrix}$$

where $f_0=a_0$, $f_1=a_0x+a_1$, $\ldots$, $f_{n-1}=a_0x^{n-1}+\ldots+a_{n-1}$, $f=a_0x^n+a_1x^{n-1}+\ldots+a_n$,

(n) $$\begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \ldots & \lambda_n \end{pmatrix} - \frac{1}{\mu}\begin{pmatrix} \lambda_1^2 & \lambda_1\lambda_2 & \ldots & \lambda_1\lambda_n \\ \lambda_2\lambda_1 & \lambda_2^2 & \ldots & \lambda_2\lambda_n \\ \cdot & \cdot & \cdot & \cdot \\ \lambda_n\lambda_1 & \lambda_n\lambda_2 & \ldots & \lambda_n^2 \end{pmatrix},$$

$$\mu = 1+\lambda_1+\lambda_2+\ldots+\lambda_n;$$

(o) $\begin{pmatrix} B^{-1}+\lambda B^{-1}UVB^{-1} & -\lambda B^{-1}U \\ -\lambda VB^{-1} & \lambda \end{pmatrix}$, $\lambda = \dfrac{1}{a-VB^{-1}U}$.

481. (a) $\begin{pmatrix} 2 & -23 \\ 0 & 8 \end{pmatrix}$, (b) $\begin{pmatrix} -3 & 2 & 0 \\ -4 & 5 & -2 \\ -5 & 3 & 0 \end{pmatrix}$,

(c) $\begin{pmatrix} 1 & -1 & -1 & 0 & \ldots & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & -1 & \ldots & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & \ldots & 0 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \ldots & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 1 & 2 \end{pmatrix}$,

(d) $\begin{pmatrix} 24 & 13 \\ -34 & -18 \end{pmatrix}$, (e) $X = E - \dfrac{n-1}{n^2}\begin{pmatrix} 1 & 1 & \ldots & 1 \\ 1 & 1 & \ldots & 1 \\ \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \ldots & 1 \end{pmatrix}$,

(f) $X = \begin{pmatrix} 1+a & b \\ -2a & 1-2b \end{pmatrix}$, (g) X does not exist.

482. It is sufficient to multiply the equation $AB=BA$ on the right and on the left by A^{-1}.

483. $\begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}$.

484. If $A^3=E$, then $|A|^3=1$, and, by virtue of its real nature, $|A|=1$. Set $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then, equating A^{-1} and A^2, we readily find that $A=E$ or $a+d=-1$, $ad-bc=1$.

485. $A=\pm E$ or $A=\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$, $a^2+bc=\pm 1$.

486. $\begin{pmatrix} a & b \\ -b & a \end{pmatrix} = aE+bI$, where $I=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Then $I^2=-E$ and, hence, the correspondence $aE+bI \to a+bi$ is an isomorphism.

487. Set $\begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix} = aE+bI+cJ+dK$,

where $I=\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $J=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $K=\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$. Then

$I^2=J^2=K^2=-E$, $IJ=-JI=K$, $JK=-KJ=I$, $KI=-IK=J$. Whence it follows that the product of two matrices of the form $a+bI+cJ+dK$ is a matrix of the same type. The same holds true for a sum and a difference so that the set of matrices at hand is a ring. Furthermore, $|A|=\begin{vmatrix} a+bi & c+di \\ -c+di & a-bi \end{vmatrix}=$ $=a^2+b^2+c^2+d^2\neq 0$ as soon as $A\neq 0$. Hence, every nonzero matrix has an inverse, and from the equality $A_1A_2=0$ (or $A_2A_1=0$) for $A_1\neq 0$, it follows that $A_2=0$. The ring of matrices under consideration is a realization of what is called the algebra of quaternions.

488. $(a_1E+b_1I+c_1J+d_1K)\,(a_2E+b_2I+c_2J+d_2K)=(a_1a_2-b_1b_2-c_1c_2-$ $-d_1d_2)E+(a_1b_2+b_1a_2+c_1d_2-d_1c_2)I+(a_1c_2-b_1d_2+c_1a_2+d_1b_2)J+(a_1d_2+$ $+b_1c_2-c_1b_2+d_1a_2)K$. Taking determinants, we get $(a_1^2+b_1^2+c_1^2+d_1^2)\,(a_2^2+b_2^2+$ $+c_2^2+d_2^2)=(a_1a_2-b_1b_2-c_1c_2-d_1d_2)^2+(a_1b_2+b_1a_2+c_1d_2-d_1c_2)^2+(a_1c_2-$ $-b_1d_2+c_1a_2+d_1b_2)^2+(a_1d_2+b_1c_2-c_1b_2+d_1a_2)^2$.

489. Interchanging two rows of a matrix is accomplished by premultiplication of the matrix

$$\begin{pmatrix} 1 & & & & & & & & \\ & \ddots & & & & & & & \\ & & 1 & & & & & & \\ & & & 0 & \dots & 1 & & & \\ & & & \vdots & 1 & \vdots & & & \\ & & & \vdots & \ddots & \vdots & & & \\ & & & & & 1 & & & \\ & & & 1 & \dots & 0 & & & \\ & & & & & & 1 & & \\ & & & & & & & \ddots & \\ & & & & & & & & 1 \end{pmatrix}.$$

Operation b is accomplished by premultiplication of the matrix

$$\begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & \dots & \alpha & & \\ & & & \ddots & \vdots & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & \vdots & \ddots & & & \\ & & \alpha & \dots & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}.$$

Operation c is accomplished by premultiplication of the matrix

$$\begin{pmatrix} 1 & & & & & & & & \\ & \cdot & & & & & & & \\ & & \cdot & & & & & & \\ & & & \cdot & & & & & \\ & & & & 1 & & & & \\ & & & & & \alpha & & & \\ & & & & & & 1 & & \\ & & & & & & & \ddots & \\ & & & & & & & & 1 \end{pmatrix}.$$

The operations a, b, c on columns are accomplished by postmultiplication of the same matrices.

490. Any matrix A can be reduced to diagonal form R by elementary transformations a, b, c on the rows and columns. Therefore, for the given matrix A there is a matrix of the form R such that $R=U_1U_2\dots U_mAV_1V_2\dots V_k$, where $U_1, \dots, U_m, V_1, \dots, V_k$ are matrices of elementary transformations. They are all nonsingular and have inverses.

Consequently, $A=PRQ$, where P and Q are nonsingular matrices.

491. By virtue of the results of Problems 489, 490, it will suffice to prove the theorem for diagonal matrices and for matrices corresponding to the operation a, because matrices corresponding to the operation b have the required form. It is easy to see that the operation a reduces to the operations b and c. Indeed, to interchange two rows, we can add the first to the second, and then from the first subtract the second, then add the first to the second and, finally, multiply the first by -1. This is equivalent to the matrix identity $E-e_{ii}-e_{kk}+$ $+e_{ik}+e_{ki}=(E-2e_{kk})(E+e_{ik})(E-e_{ki})(E+e_{ik})$. The theorem is obvious for diagonal matrices:

$$a_1e_{11}+a_2e_{22}+\dots+a_ne_{nn}$$
$$=\big(E+(a_1-1)e_{11}\big)\big(E+(a_2-1)e_{22}\big)\dots\big(E+(a_n-1)e_{nn}\big).$$

492. Let $A=P_1R_1Q_1$, $B=P_2R_2Q_2$, where P_1, Q_1, P_2, Q_2 are nonsingular matrices and R_1 and R_2 are matrices having, respectively, r_1 and r_2 units on the principal diagonal and zero elsewhere. Then $AB=P_1R_1Q_1P_2R_2Q_2$ and the rank of AB is equal to the rank of R_1CR_2, where $C=Q_1P_2$ is a nonsingular matrix. The matrix R_1CR_2 is obtained from the matrix C by replacing all elements of the last $n-r_1$ rows and $n-r_2$ columns by zeros. Since striking out one row or one column reduces the rank of a matrix by no more than unity, the rank of R_1CR_2 is not less than $n-(n-r_1)-(n-r_2)=r_1+r_2-n$.

493. It follows directly from the proportionality of all rows of a matrix of rank 1.

494. On the basis of the results of Problem 492, the rank of the desired matrix A is equal to 1 or 0. Hence

$$A=\begin{pmatrix} \lambda_1\mu_1 & \lambda_1\mu_2 & \lambda_1\mu_3 \\ \lambda_2\mu_1 & \lambda_2\mu_2 & \lambda_2\mu_3 \\ \lambda_3\mu_1 & \lambda_3\mu_2 & \lambda_3\mu_3 \end{pmatrix}.$$

Direct multiplication yields

$$0=A^2=(\lambda_1\mu_1+\lambda_2\mu_2+\lambda_3\mu_3)A$$

whence it follows that $\lambda_1\mu_1+\lambda_2\mu_2+\lambda_3\mu_3=0$.

495. Let A yield a solution to the problem different from the trivial $A=\pm E$. Then one of the matrices $A-E$ or $A+E$ has rank 1. Let

$$A+E=\begin{pmatrix} \lambda_1\mu_1 & \lambda_1\mu_2 & \lambda_1\mu_3 \\ \lambda_2\mu_1 & \lambda_2\mu_2 & \lambda_2\mu_3 \\ \lambda_3\mu_1 & \lambda_3\mu_2 & \lambda_3\mu_3 \end{pmatrix}=B.$$

Then $A^2=E-2B+B^2=E+(\lambda_1\mu_1+\lambda_2\mu_2+\lambda_3\mu_3-2)\,B$, whence it follows that for $A^2=E$ it is necessary and sufficient that the condition $\lambda_1\mu_1+\lambda_2\mu_2+\lambda_3\mu_3=2$ be fulfilled. The second case is considered in similar fashion.

496. Let there be adjoined to the matrix (A, B), where

$$A=\begin{pmatrix} a_{11} & \dots & a_{1k} \\ \cdot & \cdot\cdot\cdot\cdot & \cdot \\ a_{m1} & \dots & a_{mk} \end{pmatrix}, \quad B=\begin{pmatrix} b_{11} & \dots & b_{1s} \\ \cdot & \cdot\cdot\cdot\cdot & \cdot \\ b_{m1} & \dots & b_{ms} \end{pmatrix},$$

the column $C=\begin{pmatrix} c_1 \\ c_2 \\ \cdot \\ \cdot \\ \cdot \\ c_m \end{pmatrix}$, the adjoining of which to the matrix B does not increase its rank. Then the system of linear equations

$$\begin{gathered} b_{11}y_1+\dots+b_{1s}y_s=c_1, \\ \cdots\cdots\cdots\cdots \\ b_{m1}y_1+\dots+b_{ms}y_s=c_m \end{gathered}$$

is consistent, but then also consistent is the system

$$\begin{gathered} a_{11}x_1+\dots+a_{1k}x_k+b_{11}y_1+\dots+b_{1s}y_s=c_1, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1+\dots+a_{mk}x_k+b_{m1}y_1+\dots+b_{ms}y_s=c_m. \end{gathered}$$

Consequently, the rank of the matrix (A, B) is equal to the rank of the matrix (A, B, C).

Now suppose that the columns of the matrix B are adjoined to the matrix A gradually, one at a time. The rank can then increase by unity by virtue of what has just been proved only when the rank of B increases. Hence, the rank of $(A, B)\leqslant$ the rank of $A+$the rank of B.

497. Let the rank of $(E+A)=r_1$, the rank of $(E-A)=r_2$. Since $(E+A)+(E-A)=2E$, $r_1+r_2\geqslant n$. On the other hand, $(E+A)(E-A)=0$, therefore $0\geqslant r_1+r_2-n$. Hence, $r_1+r_2=n$.

498. The rank of the matrix $(E+A, E-A)$ is equal to n. From this matrix choose a nonsingular square matrix P of order n and let its first r columns belong to the matrix $E+A$, and let the other $n-r$ columns belong to $E-A$. Then, by virtue of the fact that $(E+A)(E-A)=0$ we have

$$(E+A)\,P=\begin{pmatrix} q_{11} & \dots & q_{1r} & 0 & \dots & 0 \\ \cdot & \cdot\cdot\cdot & \cdot & \cdot & \cdot\cdot\cdot & \cdot \\ q_{n1} & \dots & q_{nr} & 0 & \dots & 0 \end{pmatrix},$$

$$(E-A)\,P=\begin{pmatrix} 0 & \dots & 0 & q_{1,r+1} & \dots & q_{1n} \\ \cdot & \cdot\cdot\cdot & \cdot & \cdot & \cdot\cdot\cdot & \cdot \\ 0 & \dots & 0 & q_{n,r+1} & \dots & q_{nn} \end{pmatrix}.$$

Combining these equations, we get

$$2P=\begin{pmatrix} q_{11} & \cdots & q_{1r} & q_{1,r+1} & \cdots & q_{1n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ q_{n1} & \cdots & q_{nr} & q_{n,r+1} & \cdots & q_{nn} \end{pmatrix}.$$

Subtracting them, we obtain

$$2AP=\begin{pmatrix} q_{11} & \cdots & q_{1r} & -q_{1,r+1} & \cdots & -q_{1n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ q_{n1} & \cdots & q_{nr} & -q_{n,r+1} & \cdots & -q_{nn} \end{pmatrix}$$

$$=2P\begin{pmatrix} 1 & & & & & & \\ & \cdot & & & & & \\ & & \cdot & & & & \\ & & & 1 & & & \\ & & & & -1 & & \\ & & & & & \cdot & \\ & & & & & & -1 \end{pmatrix}.$$

Whence follows immediately what we sought to prove.

499. If $AA^{-1}=E$ and both matrices are integral, then $|A|\times|A^{-1}|=1$, whence it follows that $|A|=\pm 1$ because $|A|$ and $|A|^{-1}$ are integers. The condition $|A|=\pm 1$ is obviously also sufficient for the matrix A^{-1} to be integral.

500. Let A be an integral nonsingular matrix. There will be nonzero elements in its first column. By multiplying certain rows of the matrix A by -1, we can make all elements of the first column nonnegative. Choose the smallest positive one and subtract its corresponding row from some other row containing a positive element in the first column. We again get a matrix with nonnegative elements in the first column, but one of them will be less than in the original matrix. Continue the process as long as possible. In a finite number of steps, we arrive at a matrix in which all elements of the first column, except a positive one, are zero. Then, by interchanging two rows, carry the nonzero element of the first column into the first row. Next, leaving the first row fixed, use the same operations to obtain a positive element on the diagonal in the second column, all elements below it being zero. Next turn to the third column, etc. The matrix will finally become triangular. Then, by adding each row an appropriate number of times to the above-lying rows (or subtracting each row from them), we reach a situation in which the elements above the principal diagonal satisfy our requirements.

The foregoing operations are equivalent to premultiplication by certain unimodular matrices, whence immediately follows the desired result.

501. Let $A=P_1R_1=P_2R_2$, where the matrices P_1R_1 and P_2R_2 satisfy the requirements of Problem 500. Then from the equation $P_2^{-1}P_1=R_2R_1^{-1}$ it follows that the integral unimodular matrix $C=P_2^{-1}P_1$ is also of triangular form.

Let

$$R_1=\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ & a_{22} & \dots & a_{2n} \\ & & \ddots & \vdots \\ & & & a_{nn} \end{pmatrix}, \qquad R_2=\begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ & b_{22} & \dots & b_{2n} \\ & & \ddots & \vdots \\ & & & b_{nn} \end{pmatrix},$$

$$C=\begin{pmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ & c_{22} & \dots & c_{2n} \\ & & \ddots & \vdots \\ & & & c_{nn} \end{pmatrix}.$$

Then from the equation $R_2=CR_1$ we first of all conclude that $b_{11}=c_{11}a_{11}$, $\dots, b_{nn}=c_{nn}a_{nn}$, whence it follows that all c_{ii} are positive. But $c_{11}c_{22}\dots c_{nn}= =|c|=\pm 1$, hence $c_{11}=c_{22}=\dots=c_{nn}=1$ and $a_{ii}=b_{ii}$.

Furthermore, $b_{12}=c_{11}a_{12}+c_{12}a_{22}=a_{12}+c_{12}a_{22}$, whence $c_{12}=\dfrac{b_{12}-a_{12}}{a_{22}}$. But $0\leqslant b_{12}<b_{22}=a_{22}$, $0\leqslant a_{12}<a_{22}$, hence, $|c_{12}|<1$, and therefore $c_{12}=0$. Thus, comparing successively (by columns) the other elements in the matrix equation $CR_1=R_2$, we come to the conclusion that all $c_{ik}=0$ for $k>i$, that is, $C=E$. Consequently, $R_1=R_2$, $P_1=P_2$. Thus, in each class there will be one and only one matrix of the form R.

The number of matrices R with given diagonal elements $d_1, d_2, \dots, d_n$ is evidently equal to $d_2d_3^2\dots d_n^{n-1}$ and so the number of matrices R with a given determinant k is equal to $F_n(k)=\Sigma\, d_2d_3^2\dots d_n^{n-1}$, where the summation sign Σ is extended over all positive integers $d_1, d_2, \dots, d_n$ satisfying the condition $d_1d_2\dots d_n=k$. If $k=ab$, $(a, b)=1$, then each factor d_i in the equation $k= =d_1d_2\dots d_n$ is uniquely factored into two factors α_i, β_i so that $\alpha_1\alpha_2\dots\alpha_n=a$, $\beta_1\beta_2\dots\beta_n=b$. Hence,

$$F_n(k)=\sum_{d_1d_2\dots d_n=k} d_2d_3^2\dots d_n^{n-1}$$

$$=\sum_{\substack{\alpha_1\alpha_2\dots\alpha_n=a\\ \beta_1\beta_2\dots\beta_n=b}} \alpha_2\alpha_3^2\dots\alpha_n^{n-1}\,\beta_2\beta_3^2\dots\beta_n^{n-1}$$

$$=\sum_{\alpha_1\alpha_2\dots\alpha_n=a} \alpha_2\alpha_3^2\dots\alpha_n^{n-1}\cdot \sum_{\beta_1\beta_2\dots\beta_n=b} \beta_2\beta_3^2\dots\beta_n^{n-1}=F_n(a)\cdot F_n(b).$$

From this we conclude that if $k=p_1^{m_1}\dots p_s^{m_s}$ is a canonical factorization of k into simple factors, then $F_n(k)=F_n(p_1^{m_1})\dots F_n(p_s^{m_s})$.

It remains to compute $F_n(p^m)$. To do this, break up the sum for the computation of $F_n(p^m)$ into two parts, in the first of which $d_n=1$ and in the second of which d_n is divisible by p, $d_n=pd'_n$. This yields the formula $F_n(p^m)=$

$=F_{n-1}(p^m)+p^{n-1}F_n(p^{m-1})$, from which we readily establish, via mathematical induction, that

$$F_n(p^m)=\frac{(p^{m+1}-1)(p^{m+2}-1)\ \dots\ (p^{m+n-1}-1)}{(p-1)(p^2-1)\ \dots\ (p^{n-1}-1)}\ .$$

502. Choose the smallest (in absolute value) nonzero element of the matrix and carry it into the upper left corner by interchanging rows and columns. Then add the first row and the first column to all other rows and columns or subtract them as many times as is needed for all elements of the first row and the first column to be less than the corner element in absolute value. Then repeat the process. It will terminate after a finite number of steps because after each step the element which arrives in the upper left-hand corner is less in absolute value than the preceding one was. However, the process can only terminate in the fact that all elements of the first row and the first column (except the corner one) will become 0. In the same fashion, transform the matrix formed by the 2nd... nth rows and columns. The matrix will finally be reduced to diagonal form. By virtue of the result of Problem 489, all the above-described transformations are equivalent to postmultiplication and premultiplication of unimodular matrices.

503. Premultiplication of the matrix A^m is equivalent to adding the second row multiplied by m to the first. Premultiplication of B^m is equivalent to adding the first row multiplied by m to the second row.

Let $U=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a given integral matrix with determinant 1. Divide a by c: $a=mc+a_1$, $0\leqslant a_1<|c|$, then divide c by a_1: $c=m_1a_1+c_2$, $0\leqslant c_2<a_1$, etc., until the division is exact. Then $A^{-m}U=U_1=\begin{pmatrix} a_1 & b_1 \\ c & d \end{pmatrix}$, $B^{-m}U_1=$ $=U_2=\begin{pmatrix} a_1 & b_1 \\ c_2 & d_2 \end{pmatrix}$, etc. We finally arrive at the matrix U_{k+1} of the form $\begin{pmatrix} a_k & b_k \\ 0 & d_{k+1} \end{pmatrix}$ or $\begin{pmatrix} 0 & b_{k+1} \\ c_k & d_k \end{pmatrix}$. Then, by virtue of the positivity of all a_k, c_k and the unimodularity of U_{k+1}, we have $a_k=d_{k+1}=1$ in the first case, and $c_k=-b_{k+1}=1$ in the second. Thus, $U_{k+1}=\begin{pmatrix} 1 & b_k \\ 0 & 1 \end{pmatrix}=A^{b_k}$ in the first case, and $U_{k+1}=\begin{pmatrix} 0 & -1 \\ 1 & d_k \end{pmatrix}=A^{-1}BA^{d_k-1}$ in the second. The proof of the theorem is complete.

504. A matrix with determinant -1 is transformed into a matrix with determinant 1 by multiplication by C. Each such matrix is a product of the powers of A and B. But $B=CAC$.

505. Let $|A|=1$, $A^2=E$, $A\neq E$. Then (Problem 498)

$$A=P\begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix}P^{-1}$$

for some nonsingular matrix P. Define matrix P so that it is integral with the smallest possible determinant. Since $A+E=P\begin{pmatrix} 2 & & \\ & 0 & \\ & & 0 \end{pmatrix}P^{-1}$, the matrix $A+$ $+E$ is of rank 1 and, hence,

$$A+E=\begin{pmatrix} \lambda_1\mu_1 & \lambda_1\mu_2 & \lambda_1\mu_3 \\ \lambda_2\mu_1 & \lambda_2\mu_2 & \lambda_2\mu_3 \\ \lambda_3\mu_1 & \lambda_3\mu_2 & \lambda_3\mu_3 \end{pmatrix}.$$

Here, $\lambda_1\mu_1+\lambda_2\mu_2+\lambda_3\mu_3=2$ (Problem 495). Since the matrix $A+E$ is integral, the numbers $\lambda_1, \lambda_2, \lambda_3$ and the numbers μ_1, μ_2, μ_3 may be taken to be integers.

Forming a system of equations for the components of the matrix P, it is easy to verify that for P we can take the matrix

$$P=\begin{pmatrix} \lambda_1 & 0 & -\delta \\ \lambda_2 & \frac{\mu_3}{\delta} & u\mu_1 \\ \lambda_3 & -\frac{\mu_2}{\delta} & v\mu_1 \end{pmatrix}$$

where δ is the greatest common divisor of μ_2, μ_3 and u, v are integers such that $u\mu_2+v\mu_3=\delta$.

The determinant of the matrix P is equal to 2.

On the basis of the result of Problem 500, $P=QR$, where Q is unimodular and R is one of the seven possible triangular matrices of determinant 2.

Consequently, $Q^{-1}AQ$ is equal to one of the seven matrices

$$R\begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix}R^{-1}.$$

Of these matrices, there are only three distinct ones, and two of them pass into one another by a transformation via the unimodular matrix. That leaves the two indicated in the hypothesis of the problem.

506. (a) $\begin{pmatrix} 9 & 3 \\ 10 & 3 \end{pmatrix}$, (b) $\begin{pmatrix} 10 & \\ & 8 \end{pmatrix}$, (c) $\begin{pmatrix} 2 & 4 & 6 \\ 1 & 2 & 3 \\ 3 & 6 & 9 \end{pmatrix}$, (d) 13.

507. 45.

508. As a result we get Euler's identity:

$$(a_1^2+b_1^2+c_1^2)\,(a_2^2+b_2^2+c_2^2)=(aa_1+bb_1+cc_1)^2 + (a_1b_2-a_2b_1)^2+(a_1c_2-a_2c_1)^2+(b_1c_2-b_2c_1)^2.$$

509. The minor made up of the elements of rows with indices $i_1, i_2, \ldots, i_m$ and of columns with indices $k_1, k_2, \ldots, k_m$ is the determinant of the product of the matrix made up of rows $i_1, i_2, \ldots, i_m$ of the first factor by a matrix composed of the columns $k_1, k_2, \ldots, k_m$ of the second factor. It is therefore equal to the sum of all possible minors of mth order made up of the rows of the first matrix with indices $i_1, i_2, \ldots, i_m$ multiplied by minors made up of the columns of the second matrix with indices $k_1, k_2, \ldots, k_m$.

510. The diagonal minor of the matrix $\bar{A}A$ is equal to the sum of the squares of all minors of the matrix A of the same order made up of the elements of the columns having the same indices as the columns of the matrix $\bar{A}A$, which columns contain the given minor. It is therefore nonnegative.

511. If all principal minors of order k of the matrix $\bar{A}A$ are 0, then, by virtue of the result of Problem 510, all minors of order k of the matrix A are equal to 0. Hence, the rank of the matrix A, and also the rank of the matrix $\bar{A}A$, is less than k.

512. The sum of all diagonal minors of order k of the matrix $\bar{A}A$ is equal to the sum of the squares of all minors of order k of the matrix A. Also equal to this number is the sum of all diagonal minors of order k of the matrix $A\bar{A}$.

513. It is obtained by applying the theorem on the determinant of a product of two matrices to the product of the matrix $\begin{pmatrix} a_1 & a_2 & \dots & a_n \\ b_1 & b_2 & \dots & b_n \end{pmatrix}$ into its transpose.

514. It is obtained by applying the theorem on the determinant of a product to

$$\begin{pmatrix} a_1 & a_2 & \dots & a_n \\ b_1 & b_2 & \dots & b_n \end{pmatrix} \cdot \begin{pmatrix} a'_1 & b'_1 \\ a'_2 & b'_2 \\ \cdot & \cdot \\ a'_n & b'_n \end{pmatrix}.$$

515. It follows directly from the identity of Problem 513. The equal sign is only possible if the rank of the matrix $\begin{pmatrix} a_1 a_2 & \dots & a_n \\ b_1 b_2 & \dots & b_n \end{pmatrix}$ is less than two, that is, if the numbers $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ are proportional.

516. It follows directly from the identity of Problem 514. The equal sign is only possible if the numbers $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ are proportional.

517. Let matrix B have m columns, and matrix C, k columns. By the Laplace theorem, $|A| = \Sigma B_i C_i$, where B_i are all possible determinants of order m constructed from the matrix B, and C_i are their cofactors, which are equal (to within sign) to the determinants of order k constructed from the matrix C. By virtue of the Bunyakovsky inequality (Problem 515), $|A|^2 \leqslant \Sigma B_i^2 \Sigma C_i^2$. But, $\Sigma B_i^2 = |\bar{B}B|$, $\Sigma C_i^2 = |\bar{C}C|$.

518. Let

$$B = \begin{pmatrix} b_{11} & \dots & b_{1m} \\ \cdot & \cdot & \cdot \\ b_{n1} & \dots & b_{nm} \end{pmatrix}, \quad C = \begin{pmatrix} c_{11} & \dots & c_{1k} \\ \cdot & \cdot & \cdot \\ c_{n1} & \dots & c_{nk} \end{pmatrix}; \quad A = (B, C).$$

The inequality being proved is trivial if $m+k>n$; for the case $m+k=n$, it is established in Problem 517. There remains the case $m+k<n$. First assume that $\sum_{i=1}^{n} b_{ij} c_{is} = 0$ for arbitrary j, s. Then

$$\bar{A}A = \begin{pmatrix} \bar{B}B & 0 \\ 0 & \bar{C}C \end{pmatrix} \text{ and, consequently, } |\bar{A}A| = |\bar{B}B| \cdot |\bar{C}C|.$$

In the general case, it suffices to solve the problem on the assumption that the rank of the matrix A is equal to $m+k$, for otherwise the inequality is trivial.

Complete the construction of the matrix A to the square matrix (A, D) so that the rank of the matrix D is equal to $n-m-k$, and the sums of the products of the elements of any column of D by elements of any column of A are 0. For example, this can be done as follows. First complete construction of A to the nonsingular square matrix $\varepsilon' = (A, D')$, which is evidently possible, and then replace all elements of the matrix D' by their cofactors in $|\varepsilon'|$. The rank of the thus constructed matrix D will be equal to the number of its columns $n-m-k$, for it is a part of the matrix made up of the cofactors of the matrix ε', which differs from the nonsingular matrix $(\bar{\varepsilon}')^{-1}$ in the sole factor $|\varepsilon'|$.

Denote (A, D) by P, (C, D) by Q. Then, by virtue of the result of Problem 517, $|\bar{P}P| \leqslant |\bar{B}B| \cdot |\bar{Q}Q|$. But $|\bar{P}P| = |\bar{A}A| \cdot |\bar{D}D|$ and $|\bar{Q}Q| = |\bar{C}C| \cdot |\bar{D}D|$. Whence it follows that since $|\bar{D}D| > 0$,

$$|\bar{A}A| \leqslant |\bar{B}B| \cdot |\bar{C}C|.$$

519. This follows directly from the result of Problem 518 as applied to the matrix $\bar{A}$.

520. The determinant of $A^* A$ is the sum of the squares of the moduli of all minors of order m of the matrix A, where m is the number of columns of A.

521. This solution is similar to the solution of Problems 517, 518. For a square matrix, the question is resolved by applying the Laplace theorem and the Bunyakovsky inequality. It is advisable to complete the rectangular matrix to a square matrix so that the sum of the products of the elements of any column of the matrix A by the conjugates of the elements of any column of the complementary matrix is equal to 0.

522. Applying the result of Problem 521 several times to the matrix A and taking, for B, a matrix consisting of one column, we get

$$|A^*A| = \|A\|^2 \leqslant \sum_{i=1}^{n} |a_{i1}|^2 \cdot \sum_{i=1}^{n} |a_{i2}|^2 \ldots \sum_{i=1}^{n} |a_{in}|^2 \leqslant n^n M^{2n}$$

whence it follows that

$$\|A\| \leqslant n^{\frac{n}{2}} M^n.$$

523. Complete the given determinant Δ to a determinant Δ_1 of order $n+1$ by adjoining on the left a column all elements of which are equal to $\frac{M}{2}$, and a row of zeros on top. Then $\Delta = \frac{2}{M}\Delta_1$. Subtract the first column from all columns of the determinant Δ_1. We get a determinant, all elements of which do not exceed $\frac{M}{2}$. Using the result of Problem 522 yields what we set out to prove.

524. The bound $n^{\frac{n}{2}} M^n$ is attained, for example, for the modulus of the determinant

$$\begin{vmatrix} 1 & 1 & \ldots & 1 \\ 1 & \varepsilon & \ldots & \varepsilon^{n-1} \\ \cdot & \cdot & \cdot & \cdot \\ 1 & \varepsilon^{n-1} & \ldots & \varepsilon^{(n-1)^2} \end{vmatrix} \text{ where } \varepsilon = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}.$$

525. Construct a matrix of order $n=2^m$ as follows. First construct the matrix $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. Then replace each element equal to 1 by the matrix

$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ and each element equal to -1 by the matrix $-\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}=$ $=\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}$. We obtain a fourth-order matrix,

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

Operating on this matrix in the same way, we obtain an eighth-order matrix, etc.

It is readily seen that for the matrices thus constructed the sums of the products of the corresponding elements of two distinct columns are zero. Consequently,

$$\bar{A}A=\begin{pmatrix} n & 0 & \dots & 0 \\ 0 & n & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & n \end{pmatrix}, \quad |\bar{A}A|=n^n, \quad \|A\|=n^{\frac{2}{n}}.$$

The following equation holds for the matrix MA:

$$\|MA\|=M^n n^{\frac{n}{2}}.$$

526. We prove that all the elements of a matrix, the absolute value of the determinant of which has a maximal value, are equal to ± 1. Indeed, if $-1<$ $<a_{ik}<1$, $\Delta\geqslant 0$ and $A_{ik}>0$, then the determinant will be increased by substituting 1 for a_{ik}, but if $\Delta\geqslant 0$ and $A_{ik}<0$, the determinant will increase due to -1 replacing a_{ik}. If $\Delta<0$, then the absolute value of the determinant will increase when a_{ik} is replaced by unity with sign opposite that of A_{ik}. Finally, if $A_{ik}=0$, then the absolute value of the determinant will not change upon a_{ik} being replaced by 1 or -1. We can say, without loss of generality, that all the elements of the first row and the first column of a maximal determinant are equal to 1; this can be achieved by multiplying the rows and columns by -1. Now subtract the first row of the maximal determinant from all other rows. The determinant then reduces to one of order $n-1$, all elements of which are 0 or -2. This latter determinant is equal to $2^{n-1}N$, where N is some integer.

527. 4 for $n=3$, 48 for $n=5$.

528. For a singular matrix A the result is trivial. Let A be a nonsingular matrix and let $\bar{A}$ be its transpose, Δ its determinant and A' its adjoint.

Then $A'=\Delta C\bar{A}^{-1}C$, where $C=\begin{pmatrix} -1 & & & & \\ & +1 & & & \\ & & -1 & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & \cdot \end{pmatrix}$; this follows directly from the rule for constructing an inverse matrix. Therefore, $|A'|=\Delta^{n-1}$ and $(A')'=\Delta^{n-1}\cdot C(\bar{A}')^{-1}C=\Delta^{n-1}\cdot\Delta^{-1}A=\Delta^{n-2}A$.

529. Let the minor of the matrix A', which is the adjoint of the nonsingular matrix A, be made up of rows with indices $i_1<i_2<\dots<i_m$ and columns with indices $k_1<k_2<\dots<k_m$. Let $i_{m+1}<i_{m+2}<\dots<i_n$ be indices of the rows

not in the minor, let $k_{m+1}<k_{m+2}<\ldots<k_n$ be indices of columns not in the minor. Multiply the minor at hand by the determinant Δ of the matrix A:

$$\Delta\cdot\begin{vmatrix} \Delta_{i_1k_1} & \ldots & \Delta_{i_1k_m} \\ \cdot & \cdot\;\cdot\;\cdot & \cdot \\ \Delta_{i_mk_1} & \ldots & \Delta_{i_mk_m} \end{vmatrix}$$

$$=(-1)^{i_1+\ldots+i_m+k_1+\ldots+k_m}\,\Delta\cdot\begin{vmatrix} A_{i_1k_1} & \ldots & A_{i_mk_1} \\ \cdot & \cdot\;\cdot\;\cdot & \cdot \\ A_{i_1k_m} & \ldots & A_{i_mk_m} \end{vmatrix}$$

$$=\begin{vmatrix} A_{i_1k_1} & \ldots & A_{i_mk_1} & A_{i_{m+1}k_1} & \ldots & A_{i_nk_1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ A_{i_1k_m} & \ldots & A_{i_mk_m} & A_{i_{m+1}k_m} & \ldots & A_{i_nk_m} \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{vmatrix}$$

$$\times\begin{vmatrix} a_{i_1k_1} & \ldots & a_{i_1k_m} & \ldots & a_{i_1k_n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{i_mk_1} & \ldots & a_{i_mk_m} & \ldots & a_{i_mk_n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{i_nk_1} & \ldots & a_{i_nk_m} & \ldots & a_{i_nk_n} \\ \Delta & & & & \\ & \ddots & & & \\ & & \Delta & & \end{vmatrix}$$

$$=\begin{vmatrix} a_{i_{m+1}k_1} & \ldots & a_{i_{m+1}k_{m+1}} & \ldots & a_{i_{m+1}k_n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{i_nk_1} & \ldots & a_{i_nk_{m+1}} & \ldots & a_{i_nk_n} \end{vmatrix}$$

$$=\Delta^m\cdot\begin{vmatrix} a_{i_{m+1}k_{m+1}} & \ldots & a_{i_{m+1}k_n} \\ \cdot & \cdot\;\cdot\;\cdot & \cdot \\ a_{i_nk_{m+1}} & \ldots & a_{i_nk_n} \end{vmatrix}$$

whence follows what we sought to prove.

530, 531. This follows directly from the theorem on the determinant of a product of two rectangular matrices.

532. It is necessary to establish alphabetical ordering of the combinations, i.e., consider the combination $i_1<i_2<\ldots<i_m$ as preceding the combination $j_1<j_2<\ldots<j_m$ if the first nonzero difference in the sequence i_1-j_1, i_2-j_2 ..., is negative. Then each minor of the triangular matrix, the indices of the columns of which form a combination preceding the combination of the indices of the rows, is equal to 0.

533. By virtue of the results of Problems 531, 491 it suffices to prove the theorem for triangular matrices. By virtue of the result of Problem 532, we have for the triangular matrix A,

$$|A'_m|=\prod_{i_1<i_2<\ldots<i_m} a_{i_1i_1}a_{i_2i_2}\ldots a_{i_mi_m}=|A|^{C_{n-1}^{m-1}}.$$

534. Properties (a) and (b) follow directly from the definition. To establish Property (c), it is convenient to denote the elements of the Kronecker product, using for suffixes not the indices of the pairs but the pairs themselves. Let

$$C=(A'\cdot A'')\times(B'\cdot B''),\quad A'\times B'=G,\quad A''\times B''=H.$$

Then

$$c_{i_1 k_1,\, i_2 k_2}$$

$$=\sum_{i=1}^{n} a'_{i_1 i}a''_{ii_2}\cdot\sum_{k=1}^{m} b'_{k_1 k}b''_{kk_2}=\sum_{i,k} a'_{i_1 i}b'_{k_1 k}a''_{ii_2}b''_{kk_2}=\sum_{i,k} g_{i_1 k_1,\, ik}\, h_{ik,\, i_2 k_2}$$

whence $C=G\cdot H$, which completes the proof.

535. The determinant of the matrix $A\times B$ does not depend on the way the pairs are numbered, because a change in the numbering results in identical interchanges of rows and columns. Furthermore,

$$A\times B=(A\times E_m)\cdot(E_n\times B).$$

Given an appropriate numbering of the pairs, the matrix $A\times E_m$ is of the form $\begin{pmatrix} A & & & \\ & A & & \\ & & \ddots & \\ & & & A\end{pmatrix}$, the matrix A being repeated m times. Consequently, the determinant of $A\times E_m$ is equal to $|A|^m$. In the same way (but with the pairs numbered differently), we see that the determinant of $E_n\times B$ is equal to $|B|^n$. Hence, $|A\times B|=|A|^m\cdot|B|^n$.

536. An element of a row with index α and of a column with index β of the matrix C_{ik} is

$$c_{(i-1)m+\alpha,\,(k-1)m+\beta}=\sum_{s=1}^{mn} a_{(i-1)m+\alpha,\,s}\; b_{s,\,(k-1)m+\beta}$$

$$=\sum_{j=1}^{n}\sum_{v=1}^{m} a_{(i-1)m+\alpha,\,(j-1)m+v}\; b_{(j-1)m+v,\,(k-1)m+\beta}.$$

But the inner sum in the last expression is an element of the matrix $A_{ij}B_{jk}$ taken from a row with index α and a column with index β. Thus,

$$C_{ik}=\sum_{j=1}^{n} A_{ij}B_{jk}.$$

537. For $n=1$ the theorem is trivial. Assume that the theorem is proved for matrices of "order" $n-1$ and prove it for matrices of "order" n.

First consider the case when A_{11} is a nonsingular matrix:

$$C=\begin{pmatrix} A_{11} & A_{12} & \dots & A_{1n}\\ A_{21} & A_{22} & \dots & A_{2n}\\ \dots & \dots & \dots & \dots\\ A_{n1} & A_{n2} & \dots & A_{nn}\end{pmatrix}.$$

Multiply matrix C on the right by matrix D, where

$$D=\begin{pmatrix} E & -A_{11}^{-1}A_{12} & \ldots & -A_{11}^{-1}A_{1n} \\ & E & & \\ & & \ddots & \\ & & & E \end{pmatrix}.$$

Then $C'=CD$ will be of the form

$$C'=\begin{pmatrix} A_{11} & 0 & \ldots & 0 \\ A_{21} & A'_{22} & \ldots & A'_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ A_{n1} & A'_{n2} & \ldots & A'_{nn} \end{pmatrix}$$

where $A'_{ik}=A_{ik}-A_{i1}A^{-1}_{11}A_{1k}$.

All submatrices of C, D and C' commute with one another. It is easy to see that when this condition is fulfilled the theorem on the determinant of a product of two matrices is also true for formal determinants.

The matrix D has a formal determinant E, the actual determinant of D is equal to 1.

Hence, $|C|=|C'|=|A_{11}|\cdot\begin{vmatrix} A'_{22} & \ldots & A'_{2n} \\ \cdot & \cdot & \cdot \\ A'_{n2} & \ldots & A'_{nn} \end{vmatrix}$

and for the formal determinant of B we will have $B=A_{11}\cdot B'$, where B' is the formal determinant of the matrix

$$\begin{pmatrix} A'_{22} & \ldots & A'_{2n} \\ \cdot & \cdot & \cdot \\ A'_{n2} & \ldots & A'_{nn} \end{pmatrix}.$$

By the induction hypothesis,

$$|B'|=\begin{vmatrix} A'_{22} & \ldots & A'_{2n} \\ \cdot & \cdot & \cdot \\ A'_{n2} & \ldots & A'_{nn} \end{vmatrix}$$

and, consequently, $|B|=|A_{11}|\cdot|B'|=|C|$, which completes the proof.

In order to get rid of the restriction $|A_{11}|\neq 0$, the following can be done. Introduce the matrix

$$C(\lambda)=\begin{pmatrix} A_{11}+\lambda E_m & A_{12} & \ldots & A_{1n} \\ A_{21} & A_{22} & \ldots & A_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ A_{n1} & A_{n2} & \ldots & A_{nn} \end{pmatrix}$$

and denote its formal determinant by $B(\lambda)$.

Since $|A_{11}+\lambda E_m|=\lambda^m+\ldots\neq 0$, $|C(\lambda)|=|B(\lambda)|$. Both these determinants are polynomials in λ. Comparing their constant terms, we obtain $|C|=|B|$. This completes the proof.

CHAPTER 5

POLYNOMIALS AND RATIONAL FUNCTIONS OF ONE VARIABLE

538. (a) $2x^6-7x^5+6x^4-3x^3-x^2-2x+1$,

(b) $x^5-x^4-4x^3+3x+1$.

539. (a) The quotient is $2x^2+3x+11$, the remainder, $25x-5$.

(b) The quotient is $\frac{3x-7}{9}$, the remainder, $\frac{-26x-2}{9}$.

540. $p=-q^2-1$, $m=q$.

541. (1) $q=p-1$, $m=0$; (2) $q=1$, $m=\pm\sqrt{2-p}$.

542. $(-1)^n \frac{(x-1)(x-2)\ldots(x-n)}{1\cdot 2\cdot 3\ldots n}$.

543. (a) $(x-1)\ (x^3-x^2+3x-3)+5$,

(b) $(x+3)\ (2x^4-6x^3+13x^2-39x+109)-327$,

(c) $(x+1+i)\ [4x^2-(3+4i)x+(-1+7i)]+8-6i$,

(d) $(x-1+2i)\ [x^2-2ix-5-2i]-9+8i$.

544. (a) 136, (b) $-1-44i$.

545. (a) $(x+1)^4-2(x+1)^3-3(x+1)^2+4(x+1)+1$,

(b) $(x-1)^5+5(x-1)^4+10(x-1)^3+10(x-1)^2+5(x-1)+1$,

(c) $(x-2)^4-18(x-2)+38$,

(d) $(x+i)^4-2i(x+i)^3-(1+i)\ (x+i)^2-5(x+i)+7+5i$,

(e) $(x+1-2i)^4-(x+1-2i)^3+2(x+1-2i)+1$.

546. (a) $\frac{1}{(x-2)^2}+\frac{6}{(x-2)^3}+\frac{11}{(x-2)^4}+\frac{7}{(x-2)^5}$,

(b) $\frac{1}{x+1}-\frac{4}{(x+1)^2}+\frac{4}{(x+1)^3}+\frac{2}{(x+1)^5}$.

547. (a) $x^4+11x^3+45x^2+81x+55$,

(b) $x^4-4x^3+6x^2+2x+8$.

548. (a) $f(2)=18$, $f'(2)=48$, $f''(2)=124$, $f'''(2)=216$, $f^{\mathrm{IV}}(2)=240$, $f^{\mathrm{V}}(2)=120$;

(b) $f(1+2i)=-12-2i$, $f'(1+2i)=-16+8i$, $f''(1+2i)=-8+30i$, $f'''(1+2i)=24+30i$, $f^{\mathrm{IV}}(1+2i)=24$.

549. (a) 3, (b) 4.

550. $a=-5$.

551. $A=3$, $B=-4$.

552. $A=n$, $B=-(n+1)$.

555. For $f(x)$ to be divisible by $(x-1)^{k+1}$, it is necessary and sufficient hat $f(1)=a_0+a_1+\ldots+a_n=0$ and $f'(x)$ be divisible by $(x-1)^k$; for this it is in turn necessary and sufficient, given the condition $f(1)=0$, that $f_1(x)=nf(x)-xf'(x)$ be divisible by $(x-1)^k$. Regarding $f_1(x)$ formally as a polynomial of degree n, we repeat the same reasoning k times.

556. a is a root of multiplicity $k+3$, where k is the multiplicity of a as a root of $f'''(x)$.

557. $3125b^2+108a^5=0,\ a\neq 0$.

558. $b=9a^2,\ 1728a^5+c^2=0$.

559. The derivative $x^{n-m-1}\,[nx^m+(n-m)a]$ does not have multiple roots other than 0.

560. Setting the greatest common divisor of m and n equal to d, $m=dm_1$, $n=dn_1$, we get the condition in the form

$$(-1)^{n_1}(n_1-m_1)^{n_1-m_1}\,m_1^{m_1}\,a^{n_1}=b^{m_1}\,n_1^{n_1}.$$

562. A nonzero root of multiplicity $k-1$ of the polynomial

$$a_1x^{p_1}+a_2x^{p_2}+\ldots+a_kx^{p_k}$$

satisfies the equations

$$\begin{gathered}a_1x^{p_1}+a_2x^{p_2}+\ldots+a_kx^{p_k}=0,\\ p_1a_1x^{p_1}+p_2a_2x^{p_2}+\ldots+p_ka_kx^{p_k}=0,\\ p_1^2a_1x^{p_1}+p_2^2a_2x^{p_2}+\ldots+p_k^2a_kx^{p_k}=0,\\ \cdots\cdots\cdots\cdots\cdots\cdots\\ p_1^{k-2}a_1x^{p_1}+p_2^{k-2}a_2x^{p_2}+\ldots+p_k^{k-2}a_kx^{p_k}=0\end{gathered}$$

whence it follows that the numbers $a_1x^{p_1}, a_2x^{p_2}, \ldots, a_kx^{p_k}$ are proportional to the cofactors of the elements of the last row of the Vandermonde determinant

$$\Delta=\begin{vmatrix}1 & 1 & 1 & \ldots & 1\\ p_1 & p_2 & p_3 & \ldots & p_k\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ p_1^{k-1} & p_2^{k-1} & p_3^{k-1} & \ldots & p_k^{k-1}\end{vmatrix}.$$

It is easy to verify that

$$\frac{\Delta}{\Delta_i}=\prod_{s\neq i}(p_i-p_s)=\varphi'(p_i).$$

From this it follows that the numbers $a_ix^{p_i}$ are inversely proportional to $\varphi'(p_i)$, i.e.

$$a_1x^{p_1}\varphi'(p_1)=a_2x^{p_2}\varphi'(p_2)=\ldots=a_kx^{p_k}\varphi'(p_k).$$

All the foregoing reasoning is invertible.

563. If $f(x)$ is divisible by $f'(x)$, then the quotient is a polynomial of degree one with leading coefficient $\frac{1}{n}$, where n is the degree of $f(x)$. Therefore, $nf(x)=(x-x_0)f'(x)$. Differentiating, we get $(n-1)f'(x)=(x-x_0)f''(x)$, and so on, whence

$$f(x)=\frac{(x-x_0)^n}{n!}\,f^{(n)}(x)=a_0(x-x_0)^n.$$

The converse is obvious.

564. A multiple root of the polynomial $f(x)=1+\frac{x}{1}+\ldots+\frac{x^n}{n!}$ must also be a root of its derivative

$$f'(x)=1+\frac{x}{1}+\ldots+\frac{x^{n-1}}{(n-1)!}=f(x)-\frac{x^n}{n!}.$$

Hence, if $f(x_0)=f'(x_0)=0$, then $x_0=0$, but 0 is not a root of $f(x)$.

565. If $f(x)=(x-x_0)^k f_1(x)$, where $f_1(x)$ is a fractional rational function which does not vanish for $x=x_0$, then direct differentiation yields

$$f(x_0)=f'(x_0)=\ldots=f^{(k-1)}(x_0)=0,\ f^{(k)}(x_0)\neq 0.$$

Conversely, if $f(x_0)=f'(x_0)=\ldots=f^{(k-1)}(x_0)=0$ and $f^{(k)}(x_0)\neq 0$, then $f(x)=$ $=(x-x_0)^k f_1(x)$, $f_1(x_0)\neq 0$ because if it were true that $f(x)=(x-x_0)^m q(x)$, $q(x_0)\neq 0$ for $m\neq k$, then the sequence of successive derivatives vanishing for $x=x_0$ would be shorter or longer.

566. The function

$$g(x)=\frac{\psi(x)}{w(x)}=f(x)-f(x_0)-\frac{f'(x_0)}{1}(x-x_0)-\ldots-\frac{f^{(n)}(x)}{n!}(x-x_0)^n$$

satisfies the condition

$$g(x_0)=g'(x_0)=\ldots=g^{(n)}(x_0)=0.$$

Consequently, $\psi(x)=(x-x_0)^{n+1}F(x)$, where $F(x)$ is a polynomial. This completes the proof.

567. If $f_1(x)f_2(x_0)-f_2(x)f_1(x_0)$ is not identically zero, then we can take it that $f_1(x_0)\neq 0$. Consider the fractional rational function $\frac{f_2(x)}{f_1(x)}-\frac{f_2(x_0)}{f_1(x_0)}$. It is not identically zero and has x_0 as a root. The multiplicity of this root is higher by unity than the multiplicity of x_0 as a root of the derivative equal to $\frac{f_1(x)f_2'(x)-f_2(x)f_1'(x)}{[f_1(x)]^2}$ whence immediately follows the truth of the assertion being proved.

568. Let x_0 be a root of multiplicity k for $[f'(x)]^2-f(x)f''(x)$. Then $f(x_0)\neq$ $\neq 0$ because otherwise x_0 would be a common root of $f(x)$ and $f'(x)$. From the preceding problem, x_0 will be a root of multiplicity $k+1$ of the polynomial $f(x)f'(x_0)-f(x_0)f'(x)$, the degree of which does not exceed n. Hence, $k+1\leqslant$ $\leqslant n$, $k\leqslant n-1$.

569. The polynomial $f(x)f'(x_0)-f(x_0)f'(x)$ must have x_0 as a root of multiplicity n, that is, it must be equal to $A(x-x_0)^n$, where A is a constant. An expansion in powers of $x-x_0$, after the substitution $x-x_0=z$, yields

$$(a_0+a_1z+a_2z^2+\ldots+a_nz^n)a_1-(a_1+2a_2z+3a_3z^2+\ldots+na_nz^{n-1})a_0=Az^n$$

and

$$a_0=f(x_0)\neq 0.$$

Whence $a_2=\frac{a_1^2}{2a_0}$, $a_3=\frac{a_1^3}{a_0^2\,3!}$, $\ldots$, $a_n=\frac{a_1^n}{a_0^{n-1}\,n!}$.

Substituting $\frac{a_1}{a_0}=\alpha$, we get

$$f(x)=a_0\left[1+\frac{\alpha(x-x_0)}{1}+\frac{\alpha^2(x-x_0)^2}{1\cdot 2}+\ldots+\frac{\alpha^n(x-x_0)^n}{n!}\right].$$

570. For example, $\delta=\frac{1}{21}$ $\left(\frac{1}{20}\text{ is not suitable}\right)$.

571. For example, $\delta=\frac{1}{25}$ $\left(\frac{1}{24}\text{ is not suitable}\right)$.

572. For example, $M=6$.

573. For example,

$$\text{(a) } x=\rho i,\ 0<\rho<\sqrt[3]{\frac{4}{3}};\quad \text{(b) } x=\rho,\ 0<\rho<\sqrt[3]{\frac{4}{3}}.$$

574. For example, (a) $x=1-\rho,\ 0<\rho<\frac{1}{8}$;

$$\text{(b) } x=1+\rho\left(\cos\frac{(2k-1)\pi}{4}+i\sin\frac{(2k-1)\pi}{4}\right),\quad \rho<\sqrt[4]{8};$$

$$\text{(c) } x=1+\rho i,\ \rho<\frac{1}{\sqrt{3}}.$$

575. Expansion of the polynomial $f(z)$ in powers of $z-i=h$ yields

$$f(z)=(2-i)\left[1+(1-i)h^3-\frac{4+2i}{5}h^4+\frac{1+3i}{5}h^5\right].$$

Setting $h=a\ (1-i)$, we get

$$f(z)=(2-i)\left[1-4a^3+4a^3\left(\frac{4+2i}{5}a-\frac{4+2i}{5}a^2\right)\right]$$

whence

$$|f(z)|\leqslant\sqrt{5}\left(|1-4a^3|+4a^3\,\frac{1}{4}\sqrt{\frac{4}{5}}\right)<\sqrt{5}$$

for $0<a<\frac{1}{2}$.

576. Representing the polynomial in the form

$$f(z)=f(z_0)\{1+r(\cos\varphi+i\sin\varphi)(z-z_0)^k[1+(z-z_0)\psi(z)]\}$$

set $z-z_0=\rho(\cos\Theta+i\sin\Theta)$, take $\Theta=\frac{2m\pi-\varphi}{k}$ and take ρ so small that $|(z-z_0)\psi(z)|<1$. Then

$$|f(z)|=|f(z_0)|\,|1+r\rho^k+r\rho^k(z-z_0)\psi(z)|>f(z_0).$$

577. The proof is like that for a polynomial, with use made of Taylor's formula for a fractional rational function (Problem 566) which should be terminated after the first term with nonzero coefficient, not counting $f(x_0)$.

578. Denote by M the greatest lower bound of $|f(z)|$ as z varies in the region under consideration.

By dividing the region into parts, we prove the existence of a point z_0, in any neighbourhood of which the greatest lower bound of $|f(z)|$ is equal to M. If necessary, cancel from the fraction the highest possible power of $z-z_0$. After the cancellation, let $f(z)=\frac{\varphi(z)}{\psi(z)}$. Then $\psi(z_0)\neq 0$, for otherwise, in a sufficiently small neighbourhood of z_0, $|f(z)|$ would be arbitrarily great and the lower bound of $|f(z)|$ could not be equal to M in a sufficiently small neighbourhood of z_0. Consequently, $f(z)$ is continuous for $z=z_0$ also by virtue of the continuity of $|f(z_0)|=M$, which completes the proof.

579. The lemma on the increase of the modulus is lacking.

580. Under the hypothesis,

$$f(a) \neq 0, \; f'(a) = \ldots = f^{(k-1)}(a) = 0, \; f^{(k)}(a) \neq 0$$

and by Taylor's formula,

$$f(z) = f(a) + \frac{f^{(k)}(a)}{k!} (z-a)^k \; [1 + \varphi(z)], \; \varphi(a) = 0.$$

Set

$$\frac{1}{f(a)} \cdot \frac{f^{(k)}(a)}{k!} = r(\cos\varphi + i \sin\varphi), \; z - a = \rho(\cos\Theta + i\sin\Theta).$$

Take ρ so small that $|\varphi(z)| < 1$, $r\rho^k < 1$. Then

$|f(z)| = |f(a)| \cdot |1 + r\rho^k[\cos(\varphi + k\Theta) + i\sin(\varphi + k\Theta)] + r\rho^k\lambda|$, where $|\lambda| < 1$.

For $\Theta = \dfrac{(2m-1)\pi - \varphi}{k}$, $m = 1, 2, \ldots, k$, $|f(z)| < |f(a)|$.

For $\Theta = \dfrac{2m\pi - \varphi}{k}$, $m = 1, 2, \ldots, k$, $|f(z)| > |f(a)|$.

Thus, as Θ varies from $\dfrac{\pi - \varphi}{k}$ to $\dfrac{\pi - \varphi}{k} + 2\pi$, the function $|f(z)| - |f(a)|$ changes sign $2k$ times. Since $|f(z)| - |f(a)|$, as a function of Θ is continuous, $|f(z)| - |f(a)|$ vanishes $2k$ times, thus completing the proof.

581. As in Problem 580, show that $\mathrm{Re}\big(f(z)\big) - \mathrm{Re}\big(f(a)\big)$ and $\mathrm{Im}\big(f(z)\big) - \mathrm{Im}\big(f(a)\big)$ for $z = \rho(\cos\Theta + i\sin\Theta)$ changes sign $2k$ times as Θ varies through 2π, provided only that ρ is sufficiently small. Setting $\dfrac{1}{k!} f^{(k)}(a) = r(\cos\varphi + i\sin\varphi)$, we obtain, by Taylor's formula, $f(z) - f(a) = r\rho^k[\cos(\varphi + k\Theta) + i\sin(\varphi + k\Theta)][1 + \varphi(z)]$, $\varphi(a) = 0$. Choosing ρ so that $|\varphi(z)| < 1$, we obtain the following, setting $\varphi(z) = \varphi_1(z) + i\varphi_2(z)$:

$$\mathrm{Re}\big(f(z)\big) - \mathrm{Re}\big(f(a)\big) = r\rho^k\,[\cos(\varphi + k\Theta)\,\big(1 + \varphi_1(z)\big) - \sin(\varphi + k\Theta)\,\varphi_2(z)],$$

$$\mathrm{Im}\big(f(z)\big) - \mathrm{Im}\,\big(f(a)\big) = r\rho^k\,[\sin(\varphi + k\Theta)\,\big(1 + \varphi_1(z)\big) + \cos(\varphi + k\Theta)\,\varphi_2(z)].$$

Putting $\varphi + k\Theta = m\pi$, $m = 0, 1, 2, \ldots, 2k$, we get

$$\mathrm{Re}\,\big(f(z)\big) - \mathrm{Re}\,\big(f(a)\big) = r\rho^k\,(-1)^m\,(1 + \varepsilon_m)$$

where ε_m is the corresponding value of $\varphi_1(z)$, $|\varepsilon_m| < 1$.

Whence it follows that $\mathrm{Re}\,\big(f(z)\big) - \mathrm{Re}\,\big(f(a)\big)$ changes sign $2m$ times as z traverses the circle $|z - a| = \rho$. A similar result is obtained for $\mathrm{Im}\,\big(f(z)\big) - \mathrm{Im}\,\big(f(a)\big)$ by putting $\varphi + k\Theta = \dfrac{\pi}{2} + m\pi$, $m = 0, 1, \ldots, 2k$.

582. (a) $(x-1)\;(x-2)\;(x-3)$;

(b) $(x-1-i)\,(x-1+i)\,(x+1-i)\,(x+1+i)$;

(c) $\left(x+1-\sqrt{\frac{\sqrt{2}+1}{2}}-i\sqrt{\frac{\sqrt{2}-1}{2}}\right)\left(x+1-\sqrt{\frac{\sqrt{2}+1}{2}}\right.$

$$\left.+i\sqrt{\frac{\sqrt{2}-1}{2}}\right)\left(x+1+\sqrt{\frac{\sqrt{2}+1}{2}}+i\sqrt{\frac{\sqrt{2}-1}{2}}\right)$$

$$\times\left(x+1+\sqrt{\frac{\sqrt{2}+1}{2}}-i\sqrt{\frac{\sqrt{2}-1}{2}}\right);$$

(d) $(x-\sqrt{3}-\sqrt{2})(x-\sqrt{3}+\sqrt{2})(x+\sqrt{3}-\sqrt{2})(x+\sqrt{3}+\sqrt{2})$.

583. (a) $2^{n-1}\prod_{k=1}^{n}\left(x-\cos\frac{(2k-1)\pi}{2n}\right)$;

(b) $2\prod_{k=1}^{n}\left(x+\frac{\sin\left(\Theta+\frac{(2k-1)\pi}{2n}\right)}{\sin\frac{(2k-1)\pi}{2n}}\right)$; (c) $\prod_{k=1}^{m}\left(x-\tan^2\frac{(2k-1)\pi}{4m}\right)$.

584. (a) $(x^2+2x+2)(x^2-2x+2)$;

(b) $(x^2+3)(x^2+3x+3)(x^2-3x+3)$;

(c) $\left(x^2+2x+1+\sqrt{2}-2(x+1)\sqrt{\frac{\sqrt{2}+1}{2}}\right)$

$$\times\left(x^2+2x+1+\sqrt{2}+2(x+1)\sqrt{\frac{\sqrt{2}+1}{2}}\right);$$

(d) $\prod_{k=0}^{n-1}\left(x^2-2\sqrt[2n]{2}\,x\cos\frac{(8k+1)\pi}{4n}+\sqrt[n]{2}\right)$;

(e) $(x^2-x\sqrt{a+2}+1)(x^2+x\sqrt{a+2}+1)$;

(f) $\prod_{k=0}^{n-1}\left(x^2-2x\cos\frac{(3k+1)2\pi}{3n}+1\right)$.

585. (a) $(x-1)^2(x-2)(x-3)(x-1-i)=x^5-(8+i)x^4+(24+7i)x^3$ $-(34+17i)x^2+(23+17i)x-(6+6i)$;

(b) $(x+1)^3(x-3)(x-4)=x^5-4x^4-6x^3+16x^2+29x+12$;

(c) $(x-i)^2(x+1+i)=x^3+(1-i)x^2+(1-2i)x-1-i$.

586. $\prod_{k=1}^{n} X_k(x)$.

587. (a) $(x-1)^2(x-2)(x-3)(x^2-2x+2)=x^6-9x^5+33x^4-65x^3$ $+74x^2-46x+12$;

(b) $(x^2-4x+13)^3=x^6-12x^5+87x^4-376x^3+1131x^2-2028x+2197$;

(c) $(x^2+1)^2(x^2+2x+2)=x^6+2x^5+4x^4+4x^3+5x^2+2x+2$.

588. (a) $(x-1)^2(x+2)$; (b) $(x+1)^2(x^2+1)$; (c) $(x-1)^3$.

589. x^d-1, where d is the greatest common divisor of m and n.

590. x^d+a^d if the numbers $\frac{m}{d}$ and $\frac{n}{d}$ are odd; 1 if at least one of them is even; d denotes the greatest common divisor of m and n.

591. (a) $(x-1)^2(x+1)$, (b) $(x-1)^3(x+1)$,
(c) x^d-1 (d is the greatest common divisor of m and n).

592. Denote $\lambda_0=\frac{u(x_0)}{v(x_0)}$ and factor $f(x)$ into linear factors: $f(x)=$ $=(x-\lambda_0)(x-\lambda_1)\ldots(x-\lambda_{k-1})$. Then $\lambda_j\neq\lambda_0$ for $j\neq 0$. Furthermore,

$$f\left(\frac{u(x)}{v(x)}\right)=\frac{1}{[v(x)]^k}\left(u(x)-\lambda_0 v(x)\right)\ \ldots\ \left(u(x)-\lambda_{k-1}v(x)\right)\cdot$$

By virtue of the hypothesis and also of the fact that $u(x_0)-\lambda_j v(x_0)=v(x_0)$ $(\lambda_0-\lambda_j)\neq 0$, the polynomial $u(x)-\lambda_0 v(x)$ has x_0 as root of multiplicity $k>1$. Hence, $u'(x)-\lambda_0 v'(x)$ has x_0 as a root of multiplicity $k-1$. Furthermore,

$$f\left(\frac{u'(x)}{v'(x)}\right)=\frac{1}{[v'(x)]^k}\left(u'(x)-\lambda_0 v'(x)\right)\ \ldots\ \left(u'(x)-\lambda_{k-1}v'(x)\right)\cdot$$

All $u'(x)-\lambda_j v'(x)$, $j\neq 0$, do not, obviously, vanish for $x=x_0$. Consequently, $f\left(\frac{u'(x)}{v'(x)}\right)$ has x_0 as a root of multiplicity $k-1$, which is what we set out to prove.

593. If w is a root of the polynomial x^2+x+1, then $w^3=1$. Hence, $w^{3m}+$ $+w^{3n+1}+w^{3p+2}=1+w+w^2=0$.

594. The root λ of the polynomial x^2-x+1 satisfies the equation $\lambda^3=-1$. Hence,

$$\lambda^{3m}-\lambda^{3n+1}+\lambda^{3p+2}=(-1)^m-(-1)^n\lambda+(-1)^p\lambda^2$$
$$=(-1)^m-(-1)^p+\lambda[(-1)^p-(-1)^n].$$

This expression can equal zero provided only that $(-1)^m=(-1)^p=(-1)^n$, that is, if m, n, p are simultaneously even or simultaneously odd.

595. $x^4+x^2+1=(x^2+x+1)(x^2-x+1)$. These factors are relatively prime, x^2+x+1 is always a divisor of $x^{3m}+x^{3n+1}+x^{3p+2}$ (Problem 593). It remains to find out when divisibility by x^2-x+1 occurs. Substitution of the root λ of this polynomial yields

$$(-1)^m+(-1)^n\lambda+(-1)^p\lambda^2=(-1)^m-(-1)^p+\lambda[(-1)^n+(-1)^p].$$

This will yield 0 provided only that $(-1)^m=(-1)^p=-(-1)^n$, that is, if the numbers m, p and $n+1$ are simultaneously even or odd.

596. If m is not divisible by 3.

597. All roots of the polynomial $x^{k-1}+x^{k-2}+\ldots+1$ are kth roots of 1. Hence, $\xi^{ka_1}+\xi^{ka_2+1}+\ldots+\xi^{ka_k+k-1}=1+\xi+\ldots+\xi^{k-1}=0$, whence follows the divisibility, since all roots of $x^{k-1}+\ldots+1$ are prime.

598. Substitution of the root w of the polynomial x^2+x+1 into $f(x)=$ $=(1+x)^m-x^m-1$ yields $(1+w)^m-w^m-1$. But $1+w=-w^2=\lambda$, which is a primitive sixth root of unity. Furthermore, $w=\lambda^2$, whence $f(w)=\lambda^m-\lambda^{2m}-1$.

For

$m=6n \quad f(w)=-1\neq 0,$

$m=6n+1 \quad f(w)=\lambda-\lambda^2-1=0,$

$m=6n+2 \quad f(w)=\lambda^2+\lambda-1\neq 0,$

$m=6n+3 \quad f(w)=-3\neq 0,$

$m=6n+4 \quad f(w)=-\lambda+\lambda^2-1\neq 0,$

$m=6n+5 \quad f(w)=-\lambda^2+\lambda-1=0.$

Divisibility of $f(x)$ by x^2+x+1 occurs when $m=6n+1$ and $m=6n+5$.

599. For $m=6n+2$ and $m=6n+4$.

600. $f(w)=m(1+w)^{m-1}-mw^{m-1}=m[\lambda^{m-1}-\lambda^{2(m-1)}]$, $f'(w)=0$ only for $m=6n+1$.

601. For $m=6n+4$.

602. No, because the first and second derivatives do not vanish at the same time.

603. For $x=k$, $1\leqslant k\leqslant n$,

$$f(k)=1-\frac{k}{1}+\frac{k(k-1)}{1\cdot 2}-\ldots+(-1)^k\frac{k(k-1)\ \ldots\ 1}{1\cdot 2\ \ldots\ k}=(1-1)^k=0.$$

Consequently, the polynomial is divisible by $(x-1)(x-2)\ldots(x-n)$. A comparison of the leading coefficients yields

$$f(x)=\frac{(-1)^n}{n!}(x-1)(x-2)\ \ldots\ (x-n).$$

604. For m relatively prime to n.

605. If $f(x^n)$ is divisible by $x-1$, then $f(1)=0$, and, hence, $f(x)$ is divisible by $x-1$, whence it follows that $f(x^n)$ is divisible by x^n-1.

606. If $F(x)=f(x^n)$ is divisible by $(x-a)^k$, then $F'(x)=f'(x^n)nx^{n-1}$ is divisible by $(x-a)^{k-1}$, whence it follows that $f'(x^n)$ is divisible by $(x-a)^{k-1}$. In the same way, $f''(x^n)$ is divisible by $(x-a)^{k-2}, \ldots, f^{(k-1)}(x^n)$ is divisible by $x-a$. From the foregoing we conclude that $f(a^n)=f'(a^n)=\ldots=f^{(k-1)}(a^n)=0$ and, hence, $f(x)$ is divisible by $(x-a^n)^k$, $f(x^n)$ is divisible by $(x^n-a^n)^k$.

607. If $F(x)=f_1(x^3)+xf_2(x^3)$ is divisible by x^2+x+1, then $F(w)=f_1(1)+ +wf_2(1)=0$ (w is a root of x^2+x+1) and $F(w^2)=f_1(1)+w^2f_2(1)=0$, whence $f_1(1)=0$, $f_2(1)=0$.

608. The polynomial $f(x)$ has no real roots of odd multiplicity, for otherwise it would change sign. Hence, $f(x)=[f_1(x)]^2 f_2(x)$, where $f_2(x)$ is a polynomial without real roots. Separate the complex roots of the polynomial f_2 into two groups, putting conjugate roots in different groups. The products of the linear factors corresponding to the roots of each group form polynomials with conjugate coefficients $\psi_1(x)+i\psi_2(x)$ and $\psi_1(x)-i\psi_2(x)$. Hence,

$$f_2(x)=\psi_1^2(x)+\psi_2^2(x) \text{ and } f(x)=(f_1\psi_1)^2+(f_1\psi_2)^2.$$

609. (a) $-x_1, -x_2, \ldots, -x_n$, (b) $\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_n}$,

(c) $x_1-a, x_2-a, \ldots, x_n-a$, (d) $bx_1, bx_2, \ldots, bx_n$.

610. One of the roots must be equal to $-\frac{p}{2}$. The desired relation is $8r=4pq-p^3$.

611. $x_1=\frac{1}{6}$, $x_2=\frac{1}{2}$, $x_3=-\frac{1}{3}$.

612. $a^3-4ab+8c=0$.

613. The relationship among the roots is preserved for arbitrary α. Taking $\alpha=-\frac{a}{4}$, we get $y^4+a'y^3+b'y^2+c'y+d'=0$, $a'=0$, $a'^3-4a'b'+8c'=0$, for the transformed equation, whence $c'=0$.

614. $a^2d=c^2$.

615. Division by x^2 yields $x^2+\frac{d}{x^2}+a\left(x+\frac{c}{ax}\right)+b=0$.

Making the substitution $x+\frac{c}{ax}=z$, we get $x^2+\frac{d}{x^2}=x^2+\frac{c}{a^2x^2}=z^2-2\frac{c}{a}$, whence, for z, we get the quadratic equation $z^2+az+b-2\frac{c}{a}=0$. Having found z, it is easy to find x (generalized reciprocal equations).

616. (a) $x=1\pm\sqrt{3},\ 1\pm i\sqrt{2}$; (b) $x=1\pm 2i,\ -2\pm i$;

(c) $x=\frac{-1\pm\sqrt{5}}{2},\ \frac{-1\pm i\sqrt{11}}{2}$; (d) $x=1\pm\sqrt{3},\ \frac{-3\pm\sqrt{17}}{2}$.

617. $\lambda=\pm 6$.

618. (1) $b=c=0$, (any a), (2) $a=-1$, $b=-1$, $c=1$.

619. (1) $a=b=c=0$; (2) $a=1$, $b=-2$, $c=0$; (3) $a=1$, $b=-1$, $c=-1$; (4) $b=\lambda$, $a=-\frac{1}{\lambda}$, $c=\frac{2-\lambda^2}{\lambda}$, where $\lambda^3-2\lambda+2=0$.

620. $\lambda=-3$.

621. $q^3+pq+q=0$. **622.** $a_1^2-2a_2$.

623. $x_i=-\frac{a_1}{n}+\frac{2i-n-1}{2}h$, $i=1, 2, \ldots, n$ where

$$h=\frac{1}{n}\sqrt{\frac{12(n-1)a_1^2-24na_2}{n^2-1}}.$$

624. If the roots formed an arithmetic progression, then, by the formula of Problem 623, they would be:

(a) $-\frac{1}{2}$, $\frac{1}{2}$, $\frac{3}{2}$; they indeed satisfy the equation;

(b) $-\frac{5}{2}$, $-\frac{3}{2}$, $-\frac{1}{2}$, $\frac{1}{2}$; they do not satisfy the equation.

625. Let $y=Ax+B$ be the equation of the desired straight line. Then the roots of the equation $x^4+ax^3+bx^2+cx+d=Ax+B$ form an arithmetic progression. We find them in accordance with Problem 623:

$$x_i=-\frac{a}{4}+\frac{2i-5}{2}h,\ i=1, 2, 3, 4$$

where

$$h=\frac{1}{2}\sqrt{\frac{9a^2-24b}{15}}=\frac{1}{2}\sqrt{\frac{3a^2-8b}{5}}.$$

Whence

$$A-c=x_1x_4(x_2+x_3)+x_2x_3(x_1+x_4)$$
$$=-\left(\frac{a^2}{16}-\frac{9}{4}h^2\right)\frac{a}{2}-\left(\frac{a^2}{16}-\frac{1}{4}h^2\right)\frac{a}{2}=\frac{a^3-4ab}{8},$$

$$d-B=x_1x_2x_3x_4=\frac{1}{1600}(36b-11a^2)(4b+a^2).$$

Consequently

$$A=\frac{a^3-4ab+8c}{8},\quad B=d-\frac{1}{1600}(36b-11a^2)(4b+a^2).$$

The intersection points will be real and noncoincident if $3a^2-8b>0$, that is, if the second derivative of $2(6x^2+3ax+b)$ changes sign as x varies along the real axis.

626. $x^4-ax^2+1=0$ where $a=\dfrac{\alpha^4+1}{\alpha^2}$.

627. $(x^2-x+1)^3-a(x^2-x)^2=0$, $a=\dfrac{(\alpha^2-\alpha+1)^3}{(\alpha^2-\alpha)^2}$.

628. $f'(x_i)=(x_i-x_1)\dots(x_i-x_{i-1})(x_i-x_{i+1})\dots(x_i-x_n)$; $f''(x_i)=2[(x_i-x_2)\dots(x_i-x_{i-1})(x_i-x_{i+1})\dots(x_i-x_n)+(x_i-x_1)(x_i-x_3)\dots(x_i-x_{i-1})(x_i-x_{i+1})\dots(x_i-x_n)+\dots+(x_i-x_1)(x_i-x_2)\dots(x_i-x_{i-1})(x_i-x_{i+1})\dots(x_i-x_{n-1})]=2f'(x_i)\sum_{\substack{s=1\\(s\neq i)}}^{n}\dfrac{1}{x_i-x_s}$ (if $x_s\neq x_i$).

629. It follows directly from Problem 628.

630. Let $x_i=x_1+(i-1)h$. Then

$$f'(x_i)=(-1)^{n-i}(i-1)!\,(n-i)!\,h^{n-1}.$$

631. (a) $x+1$, (b) x^2+1, (c) x^3+1, (d) x^2-2x+2, (e) x^3-x+1, (f) $x+3$, (g) x^2+x+1, (h) $x^2-2x\sqrt{2}-1$, (i) $x+2$, (j) 1, (k) $2x^2+x-1$, (l) x^2+x+1.

632. (a) $(-x-1)f_1(x)+(x+2)f_2(x)=x^2-2$,

(b) $-f_1(x)+(x+1)f_2(x)=x^3+1$,

(c) $(3-x)f_1(x)+(x^2-4x+4)f_2(x)=x^2+5$,

(d) $(1-x^2)f_1(x)+(x^3+2x^2-x-1)f_2(x)=x^3+2$,

(e) $(-x^2+x+1)f_1(x)+(x^3+2x^2-5x-4)f_2(x)=3x+2$,

(f) $-\dfrac{x-1}{3}f_1(x)+\dfrac{2x^2-2x-3}{3}f_2(x)=x-1$.

633. (a) $M_2(x)=x$, $\quad M_1(x)=-3x^2-x+1$;

(b) $M_2(x)=-x-1$, $\quad M_1(x)=x^3+x^2-3x-2$;

(c) $M_2(x)=\dfrac{-x^2+3}{2}$, $\quad M_1(x)=\dfrac{x^4-2x^2-2}{2}$;

(d) $M_2(x)=-\dfrac{2x^2+3x}{6}$, $\quad M_1(x)=\dfrac{2x^3+5x^2-6}{6}$;

(e) $M_2(x)=3x^2+x-1,\ M_1(x)=-3x^3+2x^2+x-2;$

(f) $M_2(x)=-x^3-3x^2-4x-2,$

$M_1(x)=x^4+6x^3+14x^2+15x+7.$

634. (a) $M_2(x)=\dfrac{-16x^2+37x+26}{3},\ M_1(x)=\dfrac{16x^3-53x^2-37x-23}{3};$

(b) $M_2(x)=4-3x,\quad M_1(x)=1+2x+3x^2;$

(c) $M_2(x)=35-84x+70x^2-20x^3,$

$M_1(x)=1+4x+10x^2+20x^3.$

635. (a) $M_1(x)=9x^2-26x-21,$

$M_2(x)=-9x^3+44x^2-39x-7;$

(b) $M_1(x)=3x^3+3x^2-7x+2,$

$M_2(x)=-3x^3-6x^2+x+2.$

636. (a) $4x^4-27x^3+66x^2-65x+24;$

(b) $-5x^7+13x^6+27x^5-130x^4+75x^3+266x^2-440x+197.$

637. $$N(x)=1+\frac{n}{1}x+\frac{n(n+1)}{1\cdot 2}x^2+\ldots+\frac{n(n+1)\ldots(n+m-2)}{1\cdot 2\ldots(m-1)}x^{m-1};$$

$$M(x)=1+\frac{m}{1}(1-x)+\frac{m(m+1)}{1\cdot 2}(1-x)^2+\ldots+\frac{m(m+1)\ldots(m+n-2)}{1\cdot 2\ldots(n-1)}(1-x)^{n-1}$$

$$=\frac{(m+1)(m+2)\ldots(m+n-1)}{(n-1)!}-\frac{m}{1}\frac{(m+2)\ldots(m+n-1)}{(n-2)!}x+\frac{m(m+1)}{1\cdot 2}\frac{(m+3)\ldots(m+n-1)}{(n-3)!}x^2-\ldots+(-1)^{n-1}\frac{m(m+1)\ldots(m+n-2)}{(n-1)!}x^{n-1}.$$

638. 1.

639. (a) $(x+1)^4(x-2)^2$, (b) $(x+1)^4(x-4)$,

(c) $(x-1)^2(x+3)^2(x-3)$, (d) $(x-2)(x^2-2x+2)^2$,

(e) $(x^3-x^3-x-2)^2$, (f) $(x^2+1)^2(x-1)^3$,

(g) $(x^4+x^3+2x^2+x+1)^2$.

640. (a) $f(x)=x+1+\dfrac{1}{24}x(x-1)(x-2)(x-3);$

(b) $f(x)=-x^4+4x^3-x^2-7x+5;$

(c) $f(x)=1+\dfrac{2}{5}(x-1)-\dfrac{1}{105}(x-1)(4x-9)$

$+\dfrac{1}{945}(x-1)(4x-9)(x-4),$

$f(2)=1\ \frac{389}{945}=1.4116\ \ldots\ (\sqrt{2}=1.4142\ \ldots);$

(d) $f(x)=x^3-9x^2+21x-8$.

641. (a) $y=-\frac{1}{3}(x-2)(x-3)(x-4)$

$$+\frac{1}{2}(x-1)(x-3)(x-4)-2(x-1)(x-2)(x-4)$$

$$+\frac{1}{2}(x-1)(x-2)(x-3)=-\frac{4}{3}x^3+10x^2-\frac{65}{3}x+15;$$

(b) $y=\frac{1}{2}[5-(1-i)x-x^2-(1+i)x^3]$.

642. $f(x)=\frac{n+1}{2}-\frac{1}{2}\sum_{k=1}^{n-1}\left(1-i\cot\frac{k\pi}{n}\right)x^k.$

Solution. $f(x)=\sum_{s=0}^{n-1}\frac{(s+1)(x^n-1)}{(x-\varepsilon_s)n\varepsilon_s^{n-1}}$

$$=\frac{1}{n}\sum_{s=0}^{n-1}\frac{(s+1)(1-x^n)}{1-x\varepsilon_s^{-1}}=\frac{1}{n}\sum_{s=0}^{n-1}\sum_{k=0}^{n-1}(s+1)x^k\varepsilon_1^{-ks}$$

$$=\frac{1}{n}\sum_{k=0}^{n-1}x^k\sum_{s=0}^{n-1}(s+1)\varepsilon_1^{-ks}=\frac{1}{n}\sum_{s=0}^{n-1}(s+1)$$

$$+\frac{1}{n}\sum_{k=1}^{n-1}x^k\sum_{s=0}^{n-1}(s+1)\varepsilon_k^{-s}=\frac{n+1}{2}-\sum_{k=1}^{n-1}\frac{x^k}{1-\varepsilon_k^{-1}}$$

$$=\frac{n+1}{2}-\frac{1}{2}\sum_{k=1}^{n-1}\left(1-i\cot\frac{k\pi}{n}\right)x^k.$$

643. $f(x)=\sum_{k=1}^{n}\frac{y_k(x^n-1)}{(x-\varepsilon_k)n\varepsilon_k^{n-1}}=\frac{1}{n}\sum_{k=1}^{n}\frac{y_k(1-x^n)}{1-x\varepsilon_k^{-1}}.$

$$f(0)=\frac{1}{n}\sum_{k=1}^{n}y_k.$$

644. Set $\varphi(x)=(x-x_1)(x-x_2)\ldots(x-x_n)$.

Let $f(x)$ be an arbitrary polynomial of degree not higher than $n-1$, let $y_1, y_2, \ldots, y_n$ be its values for $x=x_1, x_2, \ldots, x_n$. Then

$$f(x_0)=\frac{y_1+y_2+\ldots+y_n}{n}=\sum_{k=1}^{n}\frac{y_k\varphi(x_0)}{\varphi'(x_i)(x_0-x_i)}.$$

Since $y_1, y_2, \ldots, y_n$ are arbitrary,

$$\frac{\varphi(x_0)}{\varphi'(x_k)(x_0-x_k)}=\frac{1}{n}.$$

We consider the polynomial

$$F(x)=n[\varphi(x_0)-\varphi(x)]-(x_0-x)\varphi'(x).$$

Its degree is less than n and it vanishes for $x=x_1, x_2, \ldots, x_n$. Hence, $F(x)=0$. Expand $\varphi(x)$ in powers of $(x-x_0)$:

$$\varphi(x)=\sum_{k=0}^{n} c_k (x-x_0)^k.$$

We have $\sum_{k=1}^{n}(n-k)c_k(x-x_0)^k=0$. Consequently, $c_1=c_2=\ldots=c_{n-1}=0$.

$$\varphi(x)=(x-x_0)^n+c_0, \quad x_i=x_0+\sqrt[n]{-c_0}.$$

645. $x^s=\sum_{i=1}^{n}\frac{x_i^s\varphi(x)}{(x-x_i)\varphi'(x_i)}$. A comparison of the coefficients of x^{n-1} yields

$$\sum_{i=1}^{n}\frac{x_i^s}{\varphi'(x_i)}=0.$$

646. $x^{n-1}=\sum_{i=1}^{n}\frac{x_i^{n-1}\varphi(x)}{(x-x_i)\varphi'(x_i)}$. A comparison of the coefficients of x^{n-1} yields

$$\sum_{i=1}^{n}\frac{x_i^{n-1}}{\varphi'(x_i)}=1.$$

647. $a_i=\frac{1}{\Delta}\sum_{k=1}^{n}y_k\Delta_{ki}$ where $\Delta=\begin{vmatrix} 1 & x_1 & \ldots & x_1^{n-1} \\ 1 & x_2 & \ldots & x_2^{n-1} \\ \cdot & \cdot & \cdot & \cdot \\ 1 & x_n & \ldots & x_n^{n-1}\end{vmatrix}$,

Δ_{ki} is the cofactor of the element of the kth row and $(i+1)$th column of the determinant Δ.

$$f(x)=\sum_{i=0}^{n-1}a_i x^i=\frac{1}{\Delta}\sum_{k=1}^{n}y_k\sum_{i=0}^{n-1}\Delta_{ki}x^i=\sum_{k=1}^{n}y_k\frac{\Delta_k}{\Delta}$$

where Δ_k is the determinant obtained from Δ by substituting $1, x, \ldots, x^{n-1}$ for the elements of the kth row.

Evaluating the determinants Δ_k and Δ as Vandermonde determinants yields

$$\frac{\Delta_k}{\Delta}=\frac{(x-x_1)\ \dots\ (x-x_{k-1})(x-x_{k+1})\ \dots\ (x-x_n)}{(x_k-x_1)\ \dots\ (x_k-x_{k-1})(x_k-x_{k+1})\ \dots\ (x_k-x_n)}$$
$$=\frac{\varphi(x)}{(x-x_k)\varphi'(x_k)}$$

where $\varphi(x)=(x-x_1)\ (x-x_2)\dots(x-x_n)$.

Whence $f(x)\ =\sum \dfrac{y_k\varphi\ (x)}{(x-x_k)\varphi'(x_k)}$, which is what we set out to prove.

648. $f(x)=1+\dfrac{x}{1!}+\dfrac{x(x-1)}{2!}+\dots+\dfrac{x(x-1)\ \dots\ (x-n+1)}{n!}$.

649. $f(x)=1+\dfrac{(a-1)x}{1}+\dfrac{(a-1)^2x(x-1)}{1\cdot 2}$
$$+\dots+\frac{(a-1)^n x(x-1)\ \dots\ (x-n+1)}{n!}.$$

650. $f(x)=1-\dfrac{2x}{1}+\dfrac{2x(2x-2)}{1\cdot 2}+\dots+\dfrac{2x(2x-2)\ \dots\ (2x-4n+2)}{(2n)!}$.

651. $f(x)=1-\dfrac{x-1}{2!}+\dfrac{(x-1)(x-2)}{3!}$
$$-\dots+(-1)^n\ \frac{(x-1)(x-2)\ \dots\ (x-n+1)}{n!}$$
$$=\frac{n!-(1-x)(2-x)\ \dots\ (n-x)}{n!\ x}.$$

652. $f(x)=\dfrac{\varphi(a)-\varphi(x)}{\varphi(a)(x-a)}$ where $\varphi(x)=(x-x_1)(x-x_2)\ \dots\ (x-x_n)$.

653. We seek $f(x)$ in the form

$$f(x)=A_0+A_1\ \frac{x-m}{1}+A_2\ \frac{(x-m)(x-m-1)}{1\cdot 2}$$
$$+\dots+A_n\ \frac{(x-m)(x-m-1)\ \dots\ (x-m-n+1)}{n!}$$

where $m, m+1, \dots, m+n$ are integral values of x for which, by hypothesis, $f(x)$ assumes integral values.

Successively setting $x=m,\ m+1,\ \dots,\ m+n$, we get equations for determining $A_0, A_1, \dots, A_n$:

$$A_0=f(m),$$
$$A_k=f(m+k)-A_0-\frac{k}{1}\ A_1-\frac{k(k-1)}{1\cdot 2}\ A_2-\dots-kA_{k-1},$$
$$k=1,\ 2,\ \dots,\ n$$

from which it follows that all coefficients A_k are integral. For integral values of x, all terms of $f(x)$ become binomial coefficients with integral factors A_k and for this reason are integers. Hence, $f(x)$ assumes integral values for integral values of x; this completes the proof.

654. The polynomial $F(x)=f(x^2)$ of degree $2n$ takes on integral values for $2n+1$ values of $x=-n, -(n-1), \ldots, -1, 0, 1, \ldots, n$ and, by virtue of the preceding problem, assumes integral values for all integral values of x.

655. (a) $\frac{1}{12(x-1)}-\frac{4}{3(x+2)}+\frac{9}{4(x+3)}$;

(b) $-\frac{1}{6(x-1)}+\frac{1}{2(x-2)}-\frac{1}{2(x-3)}+\frac{1}{6(x-4)}$;

(c) $\frac{2}{x-1}+\frac{-2+i}{2(x-i)}+\frac{-2-i}{2(x+i)}$;

(d) $\frac{1}{4(x-1)}-\frac{1}{4(x+1)}-\frac{i}{4(x-i)}+\frac{i}{4(x+i)}$;

(e) $\frac{1}{3}\left(\frac{1}{x-1}+\frac{\varepsilon}{x-\varepsilon}+\frac{\varepsilon^2}{x-\varepsilon^2}\right)$, $\varepsilon=-\frac{1}{2}+\frac{i\sqrt{3}}{2}$;

(f) $-\frac{1}{16}\left(\frac{1+i}{x-1-i}+\frac{1-i}{x-1+i}+\frac{-1+i}{x+1-i}+\frac{-1-i}{x+1+i}\right)$;

(g) $\sum_{k=0}^{n-1}\frac{\varepsilon_k}{x-\varepsilon_k}$, $\varepsilon_k=\cos\frac{2k\pi}{n}+i\sin\frac{2k\pi}{n}$;

(h) $-\frac{1}{n}\sum_{k=1}^{n}\frac{\eta_k}{x-\eta_k}$, $\eta_k=\cos\frac{(2k-1)\pi}{n}+i\sin\frac{(2k-1)\pi}{n}$;

(i) $\sum_{k=0}^{n}\frac{C_n^k(-1)^{n-k}}{x-k}$; (j) $\sum_{k=-n}^{n}\frac{(-1)^{n-k}C_{2n}^{n+k}}{x-k}$;

(k) $\frac{1}{n}\sum_{k=1}^{n}\frac{(-1)^{k-1}\sin\frac{2k-1}{2n}\pi}{x-\cos\frac{2k-1}{2n}\pi}$.

656. (a) $\frac{1}{3(x-1)}-\frac{x+2}{3(x^2+x+1)}$;

(b) $\frac{1}{8(x-2)}-\frac{1}{8(x+2)}+\frac{1}{2(x^2+4)}$;

(c) $\frac{1}{8}\frac{x+2}{x^2+2x+2}-\frac{1}{8}\frac{x-2}{x^2-2x+2}$;

(d) $\frac{1}{18}\left(\frac{1}{x^2+3x+3}+\frac{1}{x^2-3x+3}-\frac{2}{x^2+3}\right)$;

(e) $\frac{1}{2n+1}\left[\frac{1}{x-1}+2\sum_{k=1}^{n}\frac{x\cos\frac{2k(m+1)\pi}{2n+1}-\cos\frac{2km\pi}{2n+1}}{x^2-2x\cos\frac{2k\pi}{2n+1}+1}\right]$;

(f) $\dfrac{(-1)^m}{2n+1}\left[\dfrac{1}{x+1}+2\sum_{k=1}^{n}\dfrac{x\cos\dfrac{2k(m+1)\pi}{2n+1}+\cos\dfrac{2km\pi}{2n+1}}{x^2+2x\cos\dfrac{2k\pi}{2n+1}+1}\right]$;

(g) $\dfrac{1}{2n}\left[\dfrac{1}{x-1}-\dfrac{1}{x+1}+2\sum_{k=1}^{n-1}\dfrac{x\cos\dfrac{k\pi}{n}-1}{x^2-2x\cos\dfrac{k\pi}{n}+1}\right]$;

(h) $\dfrac{1}{n}\sum_{k=1}^{n}\dfrac{\cos\dfrac{(2k-1)m\pi}{n}-x\cos\dfrac{(2k-1)(2m+1)\pi}{2n}}{x^2-2x\cos\dfrac{(2k-1)\pi}{2n}+1}$;

(i) $\dfrac{1}{(n!)^2x}+2\sum_{k=1}^{n}\dfrac{(-1)^k x}{(n+k)!\,(n-k)!\,(x^2+k^2)}$.

657. (a) $\dfrac{1}{4(x-1)^2}-\dfrac{1}{4(x+1)^2}$;

(b) $\dfrac{1}{4(x+1)}-\dfrac{1}{4(x-1)}+\dfrac{1}{4(x-1)^2}+\dfrac{1}{4(x+1)^2}$;

(c) $\dfrac{3}{(x-1)^3}-\dfrac{4}{(x-1)^2}+\dfrac{1}{x-1}-\dfrac{1}{(x+1)^2}-\dfrac{2}{x+1}+\dfrac{1}{x-2}$;

(d) $\dfrac{1}{n^2}\left[\sum_{k=0}^{n-1}\dfrac{\varepsilon_k^2}{(x-\varepsilon_k)^2}-(n-1)\sum_{k=0}^{n-1}\dfrac{\varepsilon_k}{x-\varepsilon_k}\right]$,

$\varepsilon_k=\cos\dfrac{2k\pi}{n}+i\sin\dfrac{2k\pi}{n}$;

(e) $$\frac{1}{x^m}+\frac{\frac{n}{1}}{x^{m-1}}+\frac{\frac{n(n+1)}{1\cdot 2}}{x^{m-2}}+\ldots+\frac{\frac{n(n+1)\ldots(n+m-2)}{1\cdot 2\ldots(m-1)}}{x}$$
$$+\frac{1}{(1-x)^n}+\frac{\frac{m}{1}}{(1-x)^{n-1}}+\frac{\frac{m(m+1)}{1\cdot 2}}{(1-x)^{n-2}}$$
$$+\ldots+\frac{\frac{m(m+1)\ldots(m+n-2)}{1\cdot 2\ldots(n-1)}}{1-x}$$

(f) $$\frac{1}{(-4a^2)^n}\sum_{k=0}^{n-1}(2a)^{n-k}\frac{n(n+1)\ldots(n+k-1)}{k!}$$
$$\times\left[\frac{1}{(a-x)^{n-k}}+\frac{1}{(a+x)^{n-k}}\right];$$

(g) $\frac{1}{(4a^2)^n} \sum_{k=0}^{n-1} (2a)^{n-k} \frac{n(n+1)\ldots(n+k-1)}{k!}$

$$\times\left[\frac{1}{(a-ix)^{n-k}}+\frac{1}{(a+ix)^{n-k}}\right];$$

(h) $\sum_{k=1}^{n} \frac{g(x_k)}{[f'(x_k)]^2(x-x_k)^2}+\sum_{k=1}^{n} \frac{g'(x_k)f'(x_k)-g(x_k)f''(x_k)}{[f'(x_k)]^3(x-x_k)}$.

658. (a) $-\frac{1}{4(x+1)}+\frac{x-1}{4(x^2+1)}+\frac{x+1}{2(x^2+1)^2}$;

(b) $-\frac{1}{x}+\frac{7}{x+1}+\frac{3}{(x+1)^2}-\frac{6x+2}{x^2+x+1}-\frac{3x+2}{(x^2+x+1)^2}$;

(c) $\frac{1}{16(x-1)^2}-\frac{3}{16(x-1)}+\frac{1}{16(x+1)^2}+\frac{3}{16(x+1)}$

$$+\frac{1}{4(x^2+1)}+\frac{1}{4(x^2+1)^2};$$

(d) $\frac{1}{4n^2}\left[\frac{1}{(x-1)^2}+\frac{1}{(x+1)^2}-\frac{2n-1}{x-1}+\frac{2n-1}{x+1}\right]$

$$+\frac{1}{n^2}\sum_{k=1}^{n-1}\frac{\sin^2\frac{k\pi}{n}\left(1-2x\cos\frac{k\pi}{n}\right)}{\left(x^2-2x\cos\frac{k\pi}{n}+1\right)^2}$$

$$+\frac{1}{n^2}\sum_{k=1}^{n-1}\frac{n-\sin^2\frac{k\pi}{n}-\left(n-\frac{1}{2}\right)x\cos\frac{k\pi}{n}}{x^2-2x\cos\frac{k\pi}{n}+1}.$$

659. (a) $\frac{\varphi'(x)}{\varphi(x)}$, (b) $\frac{x\varphi'(x)-n\varphi(x)}{\varphi(x)}$, (c) $\frac{[\varphi'(x)]^2-\varphi(x)\varphi''(x)}{[\varphi(x)]^2}$.

660. (a) 9, (b) $-\frac{\varphi'(2)}{\varphi(2)}+\frac{\varphi'(1)}{\varphi(1)}=-\frac{17}{5}$, (c) 17.

661. $0.51x+2.04$. **662.** $y=\frac{1}{7}[0.55x^2+2.35x+6.98]$.

663. Substituting $\frac{p}{q}$ into $f(x)$, we obtain, after multiplication by q^n,

$a_0p^n+a_1p^{n-1}q+\ldots+a_{n-1}pq^{n-1}+a_nq^n=0,$

whence

$$\frac{a_0p^n}{q}=-(a_1p^{n-1}+\ldots+a_{n-1}pq^{n-2}+a_nq^{n-1}),$$

$$\frac{a_nq^n}{p}=-(a_0p^{n-1}+a_1p^{n-2}q+\ldots+a_{n-1}q^{n-1}).$$

The right sides of these equations contain integers. The numbers p and q are relatively prime. Hence, a_0 is divisible by q, and a_n is divisible by p.

Now arrange $f(x)$ in powers of $x-m$:

$$f(x)=a_0(x-m)^n+c_1(x-m)^{n-1}+\ldots+c_{n-1}(x-m)+c_n.$$

The coefficients $c_1, c_2, \ldots, c_n$ are integers, since m is an integer, $c_n=f(m)$. Substituting $x=\frac{p}{q}$, we get

$$a_0(p-mq)^n+c_1(p-mq)^{n-1}q+\ldots+c_{n-1}(p-mq)q^{n-1}+c_nq^n=0$$

whence we conclude that $\frac{c_n q^n}{p-mq}$ is an integer.

Since the fraction $\frac{p-mq}{q}=\frac{p}{q}-m$ is in lowest terms, the numbers $p-mq$ and q are relatively prime. Hence, $c_n=f(m)$ is divisible by $p-mq$, which is what we set out to prove.

664. We give a detailed solution of (a).

Possible values for p: $1, -1, 2, -2, 7, -7, 14, -14$. Only 1 for q (we take the sign to be attached to the numerator).

$f(1)=-4$. Hence, $p-1$ must be a divisor of 4. We reject the possibilities $p=1, -2, 7, -7, 14, -14$. It remains to test -1 and 2.

$f(-1)\neq 0$, $f(2)=0$. The only rational root is $x_1=2$.

(b) $x_1=-3$; (c) $x_1=-2$, $x_2=3$; (d) $x_1=-3$, $x_2=\frac{1}{2}$;

(e) $\frac{5}{2}$, $-\frac{3}{4}$; (f) $1, -2, 3$; (g) $\frac{1}{2}$, $-\frac{2}{3}$, $\frac{3}{4}$;

(h) no rational roots; (i) $-1, -2, -3, +4$;

(j) $\frac{1}{2}$; (k) $x_1=x_2=-\frac{1}{2}$; (l) $x_1=x_2=1$, $x_3=x_4=-3$; (m) $x_1=3$, $x_2=x_3=x_4=x_5=-1$; (n) $x_1=x_2=x_3=2$.

665. According to Problem 663, p and $p-q$ are odd at the same time. Hence, q is even and cannot equal unity.

666. By Problem 663, $p-x_1q=\pm 1$, $p-x_2q=\pm 1$, whence $(x_2-x_1)q=\pm 2$ or 0. The value 0 is dropped because $q>0$, $x_2\neq x_1$. Putting $x_2>x_1$ for definiteness, we get $(x_2-x_1)q=2$. This equation is impossible for $x_2-x_1>2$. Now put $x_2-x_1=1$ or 2. The only possible values for p and q, for which equation $(x_2-x_1)q=2$ is possible, are $p=x_1q+1$, $q=\frac{2}{x_2-x_1}$, whence the sole possibility for a rational root $\frac{p}{q}=x_1+\frac{1}{q}=\frac{x_1+x_2}{2}$. The proof is complete.

667. The Eisenstein criterion holds:

(a) for $p=2$, (b) for $p=3$, (c) for $p=3$ after expanding the polynomial in powers of $x-1$.

668. $X_p(x)=(x-1)^{p-1}+\frac{p}{1}(x-1)^{p-2}+\frac{p(p-1)}{1\cdot 2}(x-1)^{p-3}+\ldots+p.$

All coefficients $C_k=\frac{p(p-1)\ldots(p-k+1)}{1\cdot 2\ldots k}$, $k\leqslant p-1$, are divisible by p because $k!\,C_k=p(p-1)\ldots(p-k+1)$ is divisible by p, and $k!$ is relatively prime to p. Thus, after the expansion of $X_p(x)$ in powers of $x-1$, the Eisenstein criterion holds for $X_p(x)$ for p prime.

669. Apply the Eisenstein criterion for the number p, setting $x=y+1$:

$$X_{p^k}(x)=\varphi(y)=\frac{(y+1)^{p^k}-1}{(y+1)^{p^{k-1}}-1}.$$

The leading coefficient of the polynomial φ is equal to 1. The constant term of $\varphi(y)$, equal to $\varphi(0)=X_{p^k}(1)=p$, is divisible by p and is not divisible by p^2. It remains to prove that the remaining coefficients are divisible by p. To do this, we prove by induction that all the coefficients of the polynomial $(y+1)^{p^n}-1$, except the leading coefficient, are divisible by p. This is true for $n=1$. Suppose it is true for the exponent p^{n-1}, that is, $(y+1)^{p^{n-1}}=y^{p^{n-1}}+1+pw_{n-1}(y)$ where $w_{n-1}(y)$ is a polynomial with integral coefficients. Then $(y+1)^{p^n}=\left(y^{p^{n-1}}+1+pw_{n-1}(y)\right)^p=(y^{p^{n-1}}+1)^p+p\psi(y)=y^{p^n}+1+pw_n(y)$; $\psi(y)$ and $w_n(y)$ are polynomials with integral coefficients. Thus,

$$\varphi(y)=\frac{y^{p^k}+pw_k(y)}{y^{p^{k-1}}+pw_{k-1}(y)}=y^{p^k-p^{k-1}}+p\,\frac{w_k(y)-y^{p^k-p^{k-1}}w_{k-1}(y)}{y^{p^{k-1}}+pw_{k-1}(y)}=y^{p^k-p^{k-1}}+p\chi(y).$$

The coefficients of the polynomial $\chi(y)$ are integral, since $\chi(y)$ is the quotient obtained in the division of polynomials with integral coefficients, and the leading coefficient of the divisor is equal to unity. Hence, all coefficients of the polynomial $\varphi(y)$, except the leading coefficient, are divisible by p. The conditions of the Eisenstein theorem are fulfilled.

670. Suppose the polynomial is reducible: $f(x)=\varphi(x)\psi(x)$.

Then both factors have integral coefficients and their degrees are greater than 1, since $f(x)$ does not have, by hypothesis, rational roots. Let

$$\varphi(x)=b_0x^k+b_1x^{k-1}+\dots+b_k,$$

$$\psi(k)=c_0x^m+c_1x^{m-1}+\dots+c_m$$

$k\geqslant 2$, $m\geqslant 2$, $k+m=n$. Since $b_kc_m=a_n$ is divisible by p and is not divisible by p^2, we can take it that b_k is divisible by p, c_m is not divisible by p.

Let b_i be the first coefficient of $\varphi(x)$ from the end that is not divisible by p, $i\geqslant 0$. Such exists since $a_0=b_0c_0$ is not divisible by p. Then $a_{m+i}=b_ic_m+b_{i+1}c_{m-1}+\dots$ is not divisible by p, since b_ic_m is not divisible by p, and b_{i+1}, b_{i+2}, ... are divisible by p. This contradicts the hypothesis because $m+i\geqslant 2$.

671. Factoring $f(x)$ into irreducible factors with integral coefficients we consider the irreducible factor $\varphi(x)$, the constant term of which is divisible by p. Such exists since a_n is divisible by p. We denote the quotient after the division of $f(x)$ by $\varphi(x)$, by $\psi(x)$. Let

$$\varphi(x)=b_0x^m+b_1x^{m-1}+\dots+b_m,$$

$$\psi(x)=c_0x^h+c_1x^{h-1}+\dots+c_h$$

and b_i be the first (from the end) coefficient of $\varphi(x)$ not divisible by p; c_h is not divisible by p since $a_n=b_mc_h$ is not divisible by p^2.

For this reason $a_{h+i}=b_ic_h+b_{i+1}c_{h-1}+\dots$ is not divisible by p, whence follows $h+i\leqslant k$. Consequently, $m\geqslant m+h+i-k=n+i-k\geqslant n-k$.

672. (a) $f(0)=1$, $f(1)=-1$, $f(-1)=-1$.

If $f(x)=\varphi(x)\psi(x)$ and the degree of $\varphi(x)\leqslant 2$, then $\varphi(0)=\pm 1$, $\varphi(1)=\pm 1$, $\varphi(-1)=\pm 1$, that is, $\varphi(x)$ is represented by one of the tables:

x	$\varphi(x)$							
-1	1	1	1	1	-1	-1	-1	-1
0	1	-1	-1	1	-1	1	1	-1
1	-1	-1	1	1	1	1	-1	-1

The last 5 tables may be omitted since the last 4 define polynomials that differ only in sign from the polynomials represented by the first four tables, and the fourth defines a polynomial identically equal to unity. The first three yield the following possibilities:

$\varphi(x)=-(x^2+x-1),\ \varphi(x)=x^2-x-1,\ \varphi(x)=2x^2-1.$

Tests by means of division yield
$f(x)=(x^2+x-1)\ (x^2-x-1).$
(b) Irreducible, (c) irreducible, (d) $(x^2-x-1)\ (x^2-2)$.

673. A reducible polynomial of degree three has a linear factor with rational coefficients and therefore has a rational root.

674. The polynomial $x^4+ax^3+bx^2+cx+d$ has no rational roots and can be factored (in case of reducibility) only into quadratic factors with integral coefficients:

$$x^4+ax^3+bx^2+cx+d=(x^2+\lambda x+m)\ (x^2+\mu x+n).$$

The number m must obviously be a divisor of d; $mn=d$. A comparison of the coefficients of x^3 and x yields

$$\lambda+\mu=a,\quad n\lambda+m\mu=c.$$

This system is indeterminate only if $m=n$, $c=am$, that is, if $c^2=a^2d$ (see Problem 614).

But if $m\neq n$, then $\lambda=\dfrac{c-am}{n-m}=\dfrac{cm-am^2}{d-m^2}$ and this completes the proof.

675. In case of reducibility, it is necessary that

$$x^5+ax^4+bx^3+cx^2+dx+e=(x^2+\lambda x+m)\ (x^3+\lambda' x^2+\lambda'' x+n).$$

The coefficients of the factors must be integral.

A comparison of the coefficients yields $mn=e$, whence it follows that m is a divisor of e. Furthermore,

$$\begin{aligned} \lambda+\lambda' &= a,\\ n\lambda+m\lambda'' &= d,\\ m+\lambda\lambda'+\lambda'' &= b,\\ n+\lambda\lambda''+m\lambda' &= c \end{aligned}$$

whence

$$m\lambda''-n\lambda'=d-an,$$
$$\lambda\,(m\lambda''-n\lambda')+m^2\lambda'-n\lambda''=cm-bn$$

and, consequently, $(d-an)\,\lambda+m^2\,\lambda'-n\lambda''=cm-bn$. Solving this equation and $\lambda+\lambda'=a$, $n\lambda+m\lambda''=d$ simultaneously, we get

$$\lambda=\frac{am^3-cm^2-dn+be}{m^3-n^2+ae-dm}$$

which completes the proof.

676. (a) $(x^2-2x+3)(x^2-x-3)$, (b) irreducible,
(c) $(x^2-x-4)(x^2+5x+3)$, (d) $(x^2-2x+2)(x^2+3x+3)$.

677. Without loss of generality, we can seek conditions under which x^4+px^2+q can be factored into quadratic factors with rational coefficients, because if the polynomial has a rational root x_1, then $-x_1$ will also be a rational root and the linear factors corresponding to it can be combined.

Let $x^4+px^2+q=(x^2+\lambda_1x+\mu_1)(x^2+\lambda_2x+\mu_2)$.

Then

$$\lambda_1+\lambda_2=0, \quad \lambda_1\mu_2+\lambda_2\mu_1=0,$$
$$\mu_1+\lambda_1\lambda_2+\mu_2=p, \quad \mu_1\mu_2=q.$$

If $\lambda_1=0$, then $\lambda_2=0$ too. In this case, for the existence of rational μ_1 and μ_2 it is necessary and sufficient that the discriminant p^2-4q be the square of a rational number.

Let $\lambda_1\neq 0$. Then $\lambda_2=-\lambda_1$, $\mu_2=\mu_1$ and furthermore

$$q=\mu_1^2, \quad 2\mu_1-p=\lambda_1^2.$$

Thus, for the reducibility of the polynomial x^4+px^2+q it is necessary and sufficient that one of the following two conditions be fulfilled:

(a) p^2-4q is the square of a rational number;

(b) q is the square of the rational number μ_1, $2\mu_1-p$ is the square of the rational number λ_1.

678. If $x^4+ax^3+bx^2+cx+d=(x^2+p_1x+q_1)(x^2+p_2x+q_2)$, then, since $p_1+p_2=a$, we can write

$$x^4+ax^3+bx^2+cx+d=\left(x^2+\frac{1}{2}ax+\frac{\lambda}{2}\right)^2-\left(\frac{p_1-p_2}{2}x+\frac{q_1-q_2}{2}\right)^2$$

where $\lambda=q_1+q_2$. Whence it follows that the auxiliary cubic equation has the rational root $\lambda=q_1+q_2$.

679. Let $f(x)=\varphi(x)\psi(x)$ and $\varphi(x)$, $\psi(x)$ have integral coefficients. Since $f(a_i)=-1$, it must be true that $\varphi(a_i)=1$, $\psi(a_i)=-1$ or $\varphi(a_i)=-1$, $\psi(a_i)=1$ and, hence, $\varphi(a_i)+\psi(a_i)=0$, $i=1, 2, \ldots, n$.

If $\varphi(x)$ and $\psi(x)$ are both nonconstants, then the degree of $\varphi(x)+\psi(x)$ is less than n, whence it follows that $\varphi(x)+\psi(x)$ is identically zero. Thus, we must have $f(x)=-[\varphi(x)]^2$. This is impossible since the leading coefficient of $f(x)$ is positive.

680. If $f(x)=\varphi(x)\psi(x)$, then $\varphi(a_i)=\psi(a_i)=\pm 1$ since $f(a_i)=1$. Hence, if φ and ψ are nonconstants, $\varphi(x)$ is identically equal to $\psi(x)$ and

$$f(x)=[\varphi(x)]^2.$$

This is only possible for even n.

Thus, the only possible factorization is

$$(x-a_1)(x-a_2)\ldots(x-a_n)+1=[\varphi(x)]^2.$$

From this we conclude (considering the leading coefficient of $\varphi(x)$ positive) that

$$\varphi(x)+1=(x-a_1)(x-a_3)\ldots(x-a_{n-1}),$$
$$\varphi(x)-1=(x-a_2)(x-a_4)\ldots(x-a_n).$$

(In order to have the permission to write these equations, we must change the ordering of the numbers $a_1, a_2, \ldots, a_n$.) And, finally,

$$(x-a_1)(x-a_3)\ldots(x-a_{n-1})-(x-a_2)(x-a_4)\ldots(x-a_n)=2.$$

Put $a_1 > a_3 > \ldots > a_{n-1}$. Substituting $x = a_{2k}$, $k = 1, 2, \ldots, \frac{n}{2}$ in the latter equation, we get

$$(a_{2k} - a_1)(a_{2k} - a_3) \ldots (a_{2k} - a_{n-1}) = 2$$

which is to say that the number 2 must be factorable in $\frac{n}{2}$ ways into $\frac{n}{2}$ integral factors arranged in increasing order. This is only possible when $\frac{n}{2} = 2$, $2 = -2 \cdot (-1) = 1 \cdot 2$, and when $\frac{n}{2} = 1$. These two possibilities lead to the two cases of reducibility of the polynomial $f(x)$ that are mentioned in the hypothesis of the problem.

681. If an nth-degree polynomial $f(x)$ is reducible for $n = 2m$ or $n = 2m+1$, then the degree of one of its factors $\varphi(x)$ does not exceed m. If $f(x)$ assumes the values ± 1 for more than $2m$ integral values of the variable, then $\varphi(x)$ also assumes the values ± 1 for the same values of the variable. Among these values, there will exist for $\varphi(x)$ more than m values equal to $+1$ or -1. But then $\varphi(x) = +1$ or -1 identically.

682. The polynomial $f(x)$ has no real roots. Hence, if it is reducible, its factors $\varphi(x)$ and $\psi(x)$ do not have real roots and therefore do not change sign for real values of x. It may be taken that $\varphi(x) > 0$, $\psi(x) > 0$ for all real values of x. Since $f(a_k) = 1$, it follows that $\varphi(a_k) = \psi(a_k) = 1$, $k = 1, 2, \ldots, n$. If the degree of $\varphi(x)$ [or $\psi(x)$] is less than n, then $\varphi(x) = 1$ [or $\psi(x) = 1$] identically. Hence, the degrees of $\varphi(x)$ and $\psi(x)$ are equal to n. Then $\varphi(x) = 1 + \alpha(x - a_1) \ldots (x - a_n)$, $\psi(x) = 1 + \beta(x - a_1) \ldots (x - a_n)$, where α and β are some integers. But then

$$f(x) = (x - a_1)^2 \ldots (x - a_n)^2 + 1 = 1 + (\alpha + \beta)(x - a_1) \ldots (x - a_n) + \alpha\beta(x - a_1)^2 \ldots (x - a_n)^2.$$

A comparison of the coefficients of x^{2n} and x^n yields a system of equations, $\alpha\beta = 1$, $\alpha + \beta = 0$, that has no integral solutions. Consequently, $f(x)$ is irreducible.

683. Let $f(x)$ assume the value 1 more than three times. Then $f(x) - 1$ has at least four integral roots, i. e.,

$$f(x) - 1 = (x - a_1)(x - a_2)(x - a_3)(x - a_4)\, h(x)$$

where a_1, a_2, a_3, a_4 and the coefficients of the polynomial $h(x)$ are integers. For integral values of x, the expression $(x - a_1)(x - a_2)(x - a_3)(x - a_4)$ is a product of distinct integers. Two of them can be equal to $+1$ and -1, the remaining two differ from ± 1. Hence, their product cannot be equal to a prime number, in particular, -2. Thus, $f(x) - 1 \neq -2$ for integral values of x and, hence, $f(x) \neq -1$.

684. Let $f(x) = \varphi(x)\,\psi(x)$. One of the factors, $\varphi(x)$, is of degree $\leqslant \frac{n}{2}$ and assumes the values ± 1 for more than $\frac{n}{2}$ integral values of x. Since $\frac{n}{2} \geqslant 6$, it follows that $+\varphi(x)$ or $-\varphi(x)$ takes on the value 1 more than three times and, by virtue of the result of Problem 683, it cannot take on the value -1. Thus, $\varphi(x)$ or $-\varphi(x)$ assumes the value $+1$ more than $\frac{n}{2}$ times, and hence, $\varphi(x)$ or $-\varphi(x)$ is identically unity. Consequently, $f(x)$ is irreducible.

Sharpening the reasoning, we can prove the validity of the result for $n \geqslant 8$.

685. Let

$$a[\varphi(x)]^2 + b\varphi(x) + 1 = \psi(x)\, w(x).$$

One of the factors is of degree $\leqslant n$; $\psi(x)$ takes on the values ± 1 for $x = a_1, a_2, \ldots, a_n$ and since $n \geqslant 7$, all these values of $\psi(x)$ must be of like sign. Hence

$$\psi(x) = \pm 1 + \alpha(x - a_1)\ (x - a_2)\ \ldots\ (x - a_n) = \pm 1 + \alpha\varphi(x).$$

If $\alpha \neq 0$, then $w(x)$ also has degree n and $w(x) = \pm 1 + \beta\varphi(x)$. But the equation

$$a[\varphi(x)]^2 + b\varphi(x) + 1 = [\pm 1 + \alpha\varphi(x)]\ [\pm 1 + \beta\varphi(x)]$$

is impossible since the polynomial $ax^2 + bx + 1$ is irreducible by hypothesis.

686. (a) $f(x) = a_0 x^n \left(1 + \dfrac{a_1}{a_0 x} + \ldots + \dfrac{a_n}{a_0 x^n}\right)$.

Let $\max \left| \dfrac{a_k}{a_0} \right| = A$. Then for $|x| > 1$

$$|f(x)| \geqslant |a_0 x^n| \left[1 - \frac{A}{|x| - 1}\right] = |a_0 x^n|\ \frac{|x| - 1 - A}{|x| - 1} > 0$$

for $|x| > 1 + A$.

(b) $\dfrac{1}{\rho^n} f(x) = a_0 \left(\dfrac{x}{\rho}\right)^n + \dfrac{a_1}{\rho}\left(\dfrac{x}{\rho}\right)^{n-1} + \dfrac{a_2}{\rho^2}\left(\dfrac{x}{\rho}\right)^{n-2} + \ldots + \dfrac{a_n}{\rho^n}$.

By virtue of (a), for all roots

$$\frac{|x|}{\rho} \leqslant 1 + \max \left| \frac{a_k}{a_0 \rho^k} \right|, \text{ whence } |x| \leqslant \rho + \max \left| \frac{a_k}{a_0 \rho^{k-1}} \right|.$$

(c) Put $\rho = \max \sqrt[k]{\left| \dfrac{a_k}{a_0} \right|}$. Then

$$\left| \frac{a_k}{a_0} \right| \leqslant \rho^k, \quad \left| \frac{a_k}{a_0 \rho^{k-1}} \right| \leqslant \rho, \quad \max \left| \frac{a_k}{a_0 \rho^{k-1}} \right| \leqslant \rho.$$

Consequently, the moduli of all roots do not exceed

$$\rho + \rho = 2\rho = 2 \max \sqrt[k]{\left| \frac{a_k}{a_0} \right|}.$$

(d) Put $\rho = \max \sqrt[k-1]{\left| \dfrac{a_k}{a_1} \right|}$. Then $|a_k| \leqslant |a_1|\, \rho^{k-1}$, $\left| \dfrac{a_k}{a_0 \rho^{k-1}} \right| \leqslant \left| \dfrac{a_1}{a_0} \right|$.

Hence, the moduli of the roots do not exceed

$$\rho + \left| \frac{a_1}{a_0} \right| = \left| \frac{a_1}{a_0} \right| + \max \sqrt[k-1]{\left| \frac{a_k}{a_1} \right|}.$$

687. Let $f(x) = a_0 x^n + a_1 x^{n-1} + \ldots + a_n$,

$$\varphi(x) = b_0 x^n - b_1 x^{n-1} - \ldots - b_n,$$

$0 < b_0 \leqslant |a_0|$, $b_1 \geqslant |a_1|$, $\ldots$, $b_n \geqslant |a_n|$. Obviously, $|f(x)| \geqslant \varphi(|x|)$.

Furthermore, $\varphi(x) = b_0 x^n \left(1 - \dfrac{b_1}{b_0 x} - \dfrac{b_2}{b_0 x^2} - \ldots - \dfrac{b_n}{b_0 x^n}\right)$.

The expression in the brackets increases from $-\infty$ to 1 for x varying from 0 to $+\infty$.

Hence, $\varphi(x)$ has a unique positive root ξ and $\varphi(x)>0$ for $x>\xi$. Because of this, for $|x|>\xi$ we have $|f(x)|\geqslant\varphi(|x|)>0$, whence it follows that the moduli of all roots of $f(x)$ do not exceed ξ.

688. (a) Let $A=\max\left|\frac{a_k}{a_0}\right|$. It is obvious that

$$|f(x)|\geqslant|a_0x^n|\left(1-\frac{A}{|x|^r}-\frac{A}{|x|^{r+1}}-\ldots-\frac{A}{|x|^n}\right)$$

whence for $|x|>1$,

$$|f(x)|>|a_0x^n|\left(1-\frac{A}{|x|^{r-1}(|x|-1)}\right)$$

$$=\frac{|a_0x^{n-r+1}|}{|x|-1}\left[|x|^{r-1}(|x|-1)-A\right]>\frac{|a_0x|^{n-r+1}}{|x|-1}\left[(|x|-1)^r-A\right].$$

For $|x|>1+\sqrt[r]{A}$ we have $|f(x)|>0$.

(b) $\frac{1}{\rho^n}f(x)=a_0\left(\frac{x}{\rho}\right)^n+\frac{a_r}{\rho^r}\left(\frac{x}{\rho}\right)^{n-r}+\ldots+\frac{a_n}{\rho^n}$.

By virtue of (a), for all roots of $f(x)$ we have

$$\left|\frac{x}{\rho}\right|<1+\sqrt[r]{\max\left|\frac{a_k}{a_0\rho^k}\right|}\ \text{whence}\ |x|<\rho+\sqrt[r]{\max\left|\frac{a_k}{a_0\rho^{k-r}}\right|}.$$

(c) Set $\rho=\max\sqrt[k-r]{\left|\frac{a_k}{a_r}\right|}$. Then $|a_k|\leqslant|a_r|\rho^{k-r}$ and the moduli of all roots of the polynomial do not exceed

$$\sqrt[r]{\left|\frac{a_r}{a_0}\right|}+\rho=\sqrt[r]{\left|\frac{a_r}{a_0}\right|}+\max\sqrt[k-r]{\left|\frac{a_k}{a_r}\right|}.$$

689. For negative roots of the polynomial the assertion is obvious. For the sake of definiteness, set $a_0>0$ and denote $\varphi(x)=a_0x^n-b_1x^{n-1}-b_2x^{n-2}-\ldots-b_n$, where $b_k=0$ for $a_k>0$, $b_k=-a_k$ for $a_k<0$. Then, for positive x, it is obvious that

$$f(x)\geqslant\varphi(x).$$

Furthermore, $\varphi(x)$ has a unique nonnegative root ξ (see Problem 687) and $\varphi(x)>0$ for $x>\xi$. Hence, for $x>\xi$, $f(x)\geqslant\varphi(x)>0$.

690. This follows directly from 688, 689, 686 (c).

692. Expanding $f(x)$ in powers of $x-a$, we get, for $x\geqslant a$,

$$f(x)=f(a)+\frac{f'(a)}{1}(x-a)+\frac{f''(a)}{1\cdot 2}(x-a)^2+\ldots+\frac{f^{(n)}(a)}{n!}(x-a)^n>0.$$

693. We obtain the upper bound of the roots by using the results of Problems 690, 692. To determine the lower bound, substitute $-x$ for x:

(a) $0<x_i<3$, (b) $0<x_i<1$, (c) $-11<x_i<11$, (d) $-6<x_i<2$.

694. (a) $f=x^3-3x-1$, $f_1=x^2-1$, $f_2=2x+1$, $f_3=+1$. Three real roots in the intervals $(-2, -1)$, $(-1, 0)$, $(1, 2)$.

(b) $f=x^3+x^2-2x-1$, $f_1=3x^2+2x-2$, $f_2=2x+1$, $f_3=+1$. Three real roots in the intervals $(-2, -1)$, $(-1, 0)$, $(1, 2)$.

(c) $f=x^3-7x+7$, $f_1=3x^2-7$, $f_2=2x-3$, $f_3=+1$. Three real roots in the intervals $(-4, -3)$, $\left(1, \frac{3}{2}\right)$, $\left(\frac{3}{2}, 2\right)$.

(d) $f=x^3-x+5$, $f_1=3x^2-1$, $f_2=2x-15$, $f_3=-1$. One real root in the interval $(-2, -1)$.

(e) $f=x^3+3x-5$, $f_1=x^2+1$. One real root in the interval $(1, 2)$.

695. (a) $f=x^4-12x^2-16x-4$, $f_1=x^3-6x-4$, $f_2=3x^2+6x+2$, $f_3=x+1$, $f_4=1$. Four real roots in the intervals $(-3, -2)$, $(-2, -1)$, $(-1, 0)$, $(4, 5)$.

(b) $f=x^4-x-1$, $f_1=4x^2-1$, $f_2=3x+4$, $f_3=+1$. Two real roots in the intervals $(-1, 0)$ and $(1, 2)$.

(c) $f=2x^4-8x^3+8x^2-1$, $f_1=x^3-3x^2+2x$, $f_2=2x^2-4x+1$, $f_3=x-1$, $f_4=1$. Four real roots in the intervals $(-1, 0)$, $(0, 1)$, $(1, 2)$, $(2, 3)$.

(d) $f=x^4+x^2-1$, $f_1=2x^3+x$, $f_2=-x^2+2$, $f_3=-x$, $f_4=-1$. Two real roots in the intervals $(-1, 0)$ and $(0, 1)$.

(e) $f=x^4+4x^3-12x+9$, $f_1=x^3+3x^2-3$, $f_2=x^2+3x-4$, $f_3=-4x+3$, $f_4=1$. There are no real roots.

696. (a) $f=x^4-2x^3-4x^2+5x+5$, $f_1=4x^3-6x^2-8x+5$, $f_2=22x^2-22x-45$, $f_3=2x-1$, $f_4=1$. Four real roots in the intervals $(1, 2)$, $(2, 3)$, $(-1, 0)$, $(-2, -1)$.

(b) $f=x^4-2x^3+x^2-2x+1$, $f_1=2x^3-3x^2+x-1$, $f_2=x^2+5x-3$, $f_3=-9x+5$, $f_4=-1$. Two real roots in the intervals $(0, 1)$, $(1, 2)$.

(c) $f=x^4-2x^3-3x^2+2x+1$, $f_1=2x^3-3x^2-3x+1$, $f_2=9x^2-3x-5$, $f_3=9x+1$, $f_4=+1$. Four real roots in the intervals $(-2, -1)$, $(-1, 0)$, $(0, 1)$, $(2, 3)$.

(d) $f=x^4-x^3+x^2-x-1$, $f_1=4x^3-3x^2+2x-1$, $f_2=-5x^2+10x+17$, $f_3=-8x-5$, $f_4=-1$. Two real roots in the intervals $(1, 2)$, $(-1, 0)$.

(e) $f=x^4-4x^3-4x^2+4x+1$, $f_1=x^3-3x^2-2x+1$, $f_2=5x^2-x-2$, $f_3=18x+1$, $f_4=+1$. Four real roots in the intervals $(-2, -1)$, $(-1, 0)$, $(0, 1)$, $(4, 5)$.

697. (a) $f=x^4-2x^3-7x^2+8x+1$, $f_1=2x^3-3x^2-7x+4$, $f_2=17x^2-17x-8$, $f_3=2x-1$, $f_4=1$. Four real roots in the intervals $(-3, -2)$, $(-1, 0)$, $(1, 2)$, $(3, 4)$.

(b) $f=x^4-4x^2+x+1$, $f_1=4x^3-8x+1$, $f_2=8x^2-3x-4$, $f_3=87x-28$, $f_4=+1$. Four real roots in the intervals $(-3, -2)$, $(-1, 0)$, $(0, 1)$, $(1, 2)$,

(c) $f=x^4-x^3-x^2-x+1$, $f_1=4x^3-3x^2-2x-1$, $f_2=11x^2+14x-15$, $f_3=-8x+7$, $f_4=-1$. Two real roots in the intervals $(0, 1)$ and $(1, 2)$.

(d) $f=x^4-4x^3+8x^2-12x+8$, $f_1=x^3-3x^2+4x-3$, $f_2=-x^2+5x-5$, $f_3=-9x+13$, $f_4=-1$. Two real roots $x_1=2$, $1<x_2<2$.

(e) $f=x^4-x^3-2x+1$, $f_1=4x^3-3x^2-2$, $f_2=3x^2+24x-14$, $f_3=-56x+31$, $f_4=-1$. Two real roots in the intervals $(0, 1)$ and $(1, 2)$.

698. (a) $f=x^4-6x^2-4x+2$, $f_1=x^3-3x-1$, $f_2=3x^2+3x-2$, $f_3=4x+5$, $f_4=1$. Four real roots in the intervals $\left(-2, -\frac{3}{2}\right)$, $\left(-\frac{3}{2}, -1\right)$, $(0, 1)$, $(2, 3)$.

(b) $f=4x^4-12x^2+8x-1$, $f_1=2x^3-3x+1$, $f_2=6x^2-6x+1$, $f_3=2x-1$, $f_4=1$. Four real roots in the intervals $(-3, -2)$, $\left(0, \frac{1}{2}\right)$, $\left(\frac{1}{2}, 1\right)$ and $(1, 2)$.

(c) $f=3x^4+12x^3+9x^2-1$, $f_1=2x^3+6x^2+3x$, $f_2=9x^2+9x+2$, $f_3=13x+8$, $f_4=1$. Four real roots in the intervals $(-4, -3)$, $\left(-1, -\frac{2}{3}\right)$, $\left(-\frac{2}{3}, -\frac{1}{2}\right)$, $(0, 1)$.

(d) $f=x^4-x^3-4x^2+4x+1$, $f_1=4x^3-3x^2-8x+4$, $f_2=7x^2-8x-4$, $f_3=4x-5$, $f_4=1$. Four real roots in the intervals $\left(1, \frac{3}{2}\right)$, $\left(\frac{3}{2}, 2\right)$, $(-2, -1)$, $(-1, 0)$.

(e) $f=9x^4-126x^2-252x-140$, $f_1=x^3-7x-7$, $f_2=9x^2+27x+20$, $f_3=2x+3$, $f_4=1$. Four real roots in the intervals $(4, 5)$, $\left(-\frac{4}{3}, -1\right)$, $\left(-\frac{5}{3}, -\frac{4}{3}\right)$, $\left(-2, -\frac{5}{3}\right)$.

699. (a) $f=2x^5-10x^3+10x-3$, $f_1=x^4-3x^2+1$, $f_2=4x^3-8x+3$, $f_3=4x^2+3x-4$, $f_4=x$, $f_5=1$. Five real roots in the intervals $\left(-2, -\frac{3}{2}\right)$, $\left(-\frac{3}{2}, -1\right)$, $\left(0, \frac{1}{2}\right)$, $\left(\frac{1}{2}, 1\right)$, $(1, 2)$.

(b) $f=x^6-3x^5-3x^4+11x^3-3x^2-3x+1$, $f_1=2x^5-5x^4-4x^3+11x^2-2x-1$, $f_2=3x^4-6x^3-x^2+4x-1$, $f_3=4x^3-6x^2+1$, $f_4=26x^2-26x+5$, $f_5=2x-1$, $f_6=1$. Six real roots in the intervals $(-2, -1)$, $(-1, 0)$, $\left(0, \frac{1}{2}\right)$, $\left(\frac{1}{2}, 1\right)$, $(1, 2)$, $(2, 3)$.

(c) $f=x^5+x^4-4x^3-3x^2+3x+1$, $f_1=5x^4+4x^3-12x^2-6x+3$, $f_2=4x^3+3x^2-6x-2$, $f_3=3x^2+2x-2$, $f_4=2x+1$, $f_5=1$. Five real roots in the intervals $\left(-2, -\frac{3}{2}\right)$, $\left(-\frac{3}{2}, -1\right)$, $(-1, 0)$, $(0, 1)$, $(1, 2)$.

(d) $f=x^5-5x^3-10x^2+2$, $f_1=x^4-3x^2-4x$, $f_2=x^3+3x^2-1$, $f_3=-2x^2+x+1$, $f_4=-3x-1$, $f_5=-1$. Three real roots in the intervals $(-1, 0)$, $(0, 1)$, $(2, 3)$.

700. (a) $f=x^4+4x^2-1$, $f_1=x$, $f_2=1$. Two real roots in the intervals $(-1, 0)$, $(0, 1)$.

(b) $f=x^4-2x^3+3x^2-9x+1$, $f_1=2x-3$, $f_2=1$. Two real roots in the intervals $(0, 1)$ and $(2, 3)$.

(c) $f=x^4-2x^3+2x^2-6x+1$, $f_1=2x-3$, $f_2=1$. Two real roots in the intervals $(0, 1)$ and $(2, 3)$.

(d) $f=x^5+5x^4+10x^2-5x-3$, $f_1=x^2+4x-1$, $f_2=5x-1$, $f_3=1$. Three real roots in the intervals $(0, 1)$, $(-1, 0)$, $(-6, -5)$.

701. The Sturm sequence is formed by the polynomials x^3+px+q, $3x^2+p$, $-2px-3q$, $-4p^3-27q^2$. If $-4p^3-27q^2>0$, then $p<0$. All leading coefficients of the Sturm polynomials are positive and so all the roots of x^3+px+q are real. If $-4p^3-27q^2<0$, then, irrespective of the sign of p, the Sturm sequence has, for $-\infty$, two changes of sign, and for $+\infty$, one change of sign. In this case, x^3+px+q has one real root.

702. The Sturm sequence is formed by the polynomials

$$x^n+px+q,\ nx^{n-1}+p,\ -(n-1)px-nq,\ -p-n\left(\frac{-nq}{(n-1)p}\right)^{n-1}.$$

For odd n, the sign of the last expression coincides with the sign of $\Delta=-(n-1)^{n-1}p^n-n^n q^{n-1}$. If $\Delta>0$, then necessarily $p<0$. In this case, the polynomial has three real roots. If $\Delta<0$, then, irrespective of the sign of p, the polynomial has one real root.

For even n, the sign of the last expression in the Sturm sequence coincides with the sign of $-p\Delta$, where $\Delta=(n-1)^{n-1}p^n-n^n q^{n-1}$. The distribution of signs in the Sturm sequence is given in the following table for various combinations of the signs of p and Δ:

			f	f_1	f_2	f_3
1. $p>0$,	$\Delta>0$	$-\infty$	+	−	+	−
		$+\infty$	+	+	−	−
2. $p<0$,	$\Delta>0$	$-\infty$	+	−	−	+
		$+\infty$	+	+	+	+
3. $p>0$,	$\Delta<0$	$-\infty$	+	−	+	+
		$+\infty$	+	+	−	+
4. $p<0$,	$\Delta<0$	$-\infty$	+	−	−	−
		$+\infty$	+	+	+	−

From this table it follows that for $\Delta>0$ the polynomial has two real roots, for $\Delta<0$ there are no real roots.

703. The Sturm sequence is formed by the polynomials $f=x^5-5ax^3+5a^2x+2b$, $f_1=x^4-3ax^2+a^2$, $f_2=ax^3-2a^2x-b$, $f_3=a(a^2x^2-bx-a^3)$, $f_4=a(a^5-b^2)x$, $f_5=1$.

If $\Delta=a^5-b^2>0$, then $a>0$, and all leading coefficients of the Sturm polynomials are positive. In this case, all five roots of the polynomial f are real. If $\Delta<0$, then, depending on the sign of a, the distribution of signs looks like this:

		f	f_1	f_2	f_3	f_4	f_5
$a>0$	$-\infty$	−	+	−	+	+	+
	$+\infty$	+	+	+	+	−	+
$a<0$	$-\infty$	−	+	+	−	−	+
	$+\infty$	+	+	−	−	+	+

Consequently, for $\Delta<0$, the polynomial f has one real root.

704. Let f_λ and $f_{\lambda+1}$ be two consecutive polynomials of a "complete" Sturm sequence. If their leading coefficients have the same signs, then their values, for $+\infty$, do not constitute a change of sign, while the values for $-\infty$ yield a change of sign, since the degree of one of the polynomials is even, while the degree of the other is odd. Now if the leading coefficients have opposite signs, then the values of f_λ and $f_{\lambda+1}$ for $+\infty$ yield a change of sign, and for $-\infty$ do not. Therefore, denoting by ν_1 and ν_2 the number of variations of sign in the Sturm sequence, for $-\infty$ and $+\infty$, we have that $\nu_1+\nu_2=n$. On the other hand, $\nu_1-\nu_2$ is equal to the number N of real roots of the polynomial. Consequently, $\nu_2=\dfrac{n-N}{2}$, which is what we set out to prove.

705. This is proved like the Sturm theorem, with the sole difference that we have to see that there is an increase (not a decrease) in the number of variations of sign per unit when passing through a root of the original polynomial.

706. The sequence of polynomials thus constructed is a Sturm sequence for the interval $x_0 \leqslant x < +\infty$ and satisfies the conditions of Problem 705 for the interval $-\infty < x \leqslant x_0$. Hence, the number of roots of f in the interval (x_0, ∞) is equal to $\nu(x_0) - \nu(+\infty)$, the number of roots of f in the interval $(-\infty, x_0)$ is equal to $\nu(x_0) - \nu(-\infty)$, where ν is the number of variations of sign of the corresponding values of the polynomials.

The total number of real roots is equal to

$$2\nu(x_0) - \nu(+\infty) - \nu(-\infty).$$

707. Applying the Euler theorem to

$$P_n = (-1)^n e^{\frac{x^2}{2}} \frac{d^n e^{-\frac{x^2}{2}}}{dx^n} = (-1)^n e^{\frac{x^2}{2}} \frac{d^{n-1}\left(-xe^{-\frac{x^2}{2}}\right)}{dx^{n-1}}$$

yields

$$P_n = (-1)^{n-1} e^{\frac{x^2}{2}} \left(x \frac{d^{n-1} e^{-\frac{x^2}{2}}}{dx^{n-1}} + (n-1) \frac{d^{n-2} e^{-\frac{x^2}{2}}}{dx^{n-2}} \right)$$

whence

$$P_n = xP_{n-1} - (n-1) P_{n-2}.$$

On the other hand, differentiating the equation defining P_{n-1}, we get

$$P'_{n-1} = (-1)^{n-1} xe^{\frac{x^2}{2}} \frac{d^{n-1} e^{-\frac{x^2}{2}}}{dx^{n-1}} + (-1)^{n-1} e^{\frac{x^2}{2}} \frac{d^n e^{-\frac{x^2}{2}}}{dx^n}$$

whence

$$P'_{n-1} = xP_{n-1} - P_n.$$

Comparing this with the previous formula, we get

$$P'_{n-1} = (n-1) P_{n-2} \text{ and so } P'_n = nP_{n-1}.$$

It follows from the derived formulas that the sequence $P_n, P_{n-1}, \ldots, P_1, P_0 = 1$ is a Sturm sequence for the polynomials P_n since P_{n-1} differs from P'_n in the factor n alone, and $P_{\lambda-1}$ is, to within a positive factor, the remainder (taken with sign reversed) after division of $P_{\lambda+1}$ by P_λ.

All the leading coefficients of the polynomials P_n are equal to $+1$. Hence, all the roots of P_n are real.

708. Differentiating the equation defining P_n, we obtain

$$P'_n = (-1)^n e^x \frac{d^n (x^n e^{-x})}{dx^n} + (-1)^n e^x \frac{d^n (nx^{n-1} e^{-x} - x^n e^{-x})}{dx^n}$$

whence

$$P'_n = (-1)^n ne^x \frac{d^n (x^{n-1} e^{-x})}{dx^n}.$$

Furthermore,

$$P_n = (-1)^n e^x \frac{d^n (x \cdot x^{n-1} e^{-x})}{dx^n}$$

$$= (-1)^n e^x \left[x \frac{d^n (x^{n-1} e^{-x})}{dx^n} + n \frac{d^{n-1} (x^{n-1} e^{-x})}{dx^{n-1}} \right] = \frac{x}{n} P'_n - nP_{n-1}$$

whence

$$xP_n'=nP_n+n^2P_{n-1}.$$

On the other hand,

$$P_n'=(-1)^n ne^x \frac{d^{n-1}[(n-1)x^{n-2}e^{-x}-x^{n-1}e^{-x}]}{dx^{n-1}}$$

whence $P_n'=-nP_{n-1}'+nP_{n-1}$. Multiplying by x and substituting in place of xP_n' and xP_{n-1}' their expression in terms of P_n, P_{n-1}, P_{n-2}, we get

$$P_n=(x-2n+1)P_{n-1}-(n-1)^2P_{n-2}.$$

From these relations it is seen that consecutive polynomials P_n do not vanish simultaneously, and if $P_{n-1}=0$, then P_n and P_{n-2} have opposite signs. Furthermore, from $\frac{P_{n-1}}{P_n}=-\frac{1}{n}+\frac{xP_n'}{n^2P_n}$ it follows that $\frac{P_{n-1}}{P_n}$ changes sign from minus to plus when going through a positive root of P_n. Thus, the sequence P_n, P_{n-1}, $\ldots$, P_1, $P_0=1$ is a Sturm sequence for P_n in the interval $(0, \infty)$. The leading coefficients of all P_n are equal to unity. $P_n(0)=(-1)^n n!$. Hence, $\nu(0)-\nu(+\infty)=n$, that is P_n has n positive roots.

709. $E_n'=E_{n-1}$. Also, $E_n=E_{n-1}-\left(-\frac{x^n}{n!}\right)$.

Therefore, the polynomials E_n, E_{n-1} and $-\frac{x^n}{n!}$ form a Sturm sequence for E_n on the interval $(-\infty, -\varepsilon)$ for arbitrarily small ε. The distribution of signs is given by the following table:

$-\infty$	$(-1)^n$	$(-1)^{n-1}$	$(-1)^{n-1}$
$-\varepsilon$	$+$	$+$	$(-1)^{n-1}$

Hence, for even n, the polynomial E_n has no negative roots, for odd n, the polynomial E_n has one negative root. Furthermore, for $x\geqslant 0$, the polynomial $E_n(x)>0$.

710. Use the Euler formula to transform the identity

$$\frac{d^{n+1}\left(x^2e^{\frac{1}{x}}\right)}{dx^{n+1}}=\frac{d^n\left[(2x-1)e^{\frac{1}{x}}\right]}{dx^n}.$$

We get

$$x^2\frac{d^{n+1}e^{\frac{1}{x}}}{dx^{n+1}}+2(n+1)x\frac{d^ne^{\frac{1}{x}}}{dx^n}+(n+1)n\frac{d^{n-1}e^{\frac{1}{x}}}{dx^{n-1}}$$

$$=(2x-1)\frac{d^ne^{\frac{1}{x}}}{dx^n}+2n\frac{d^{n-1}e^{\frac{1}{x}}}{dx^{n-1}}$$

whence $P_n=(2nx+1)P_{n-1}-n(n-1)P_{n-2}x^2$. On the other hand, by differentiating the equation defining P_{n-1}, we get

$$P_n=(2nx+1)P_{n-1}-x^2P_{n-1}'.$$

Comparing the results, we see that $P_{n-1}'=n(n-1)P_{n-2}$ and, hence, $P_n'=(n+1)nP_{n-1}$. By virtue of the established relations, the sequence of polynomials $P_n, P_{n-1}, P_{n-2}, \ldots, P_0=1$ forms a Sturm sequence for P_n. The leading coefficients of all P_n are positive. Consequently, all the roots of P_n are real.

711. Computing

$$\frac{d^n\left(\frac{x^2}{x^2+1}\right)}{dx^n}=-\frac{d^n\left(\frac{1}{x^2+1}\right)}{dx^n}$$

by two methods, we obtain

$$P_n-2x\,P_{n-1}+(x^2+1)\,P_{n-2}=0.$$

Differentiation of the equation defining P_{n-1} yields $P_n=2xP_{n-1}-\frac{x^2+1}{n}$ P'_{n-1} whence $P'_{n-1}=nP_{n-2}$ and, hence, $P'_n=(n+1)\,P_{n-1}$.

It follows from the derived relations that $P_n, P_{n-1}, \ldots, P_0=1$ form a Sturm sequence for P_n. All the leading coefficients of the sequence are positive and so all the roots of P_n are real.

The solution of this problem is straight forward. Namely,

$$\frac{1}{x^2+1}=\frac{1}{2i}\left(\frac{1}{x-i}+\frac{1}{x+i}\right)$$

whence we find that

$$P_n(x)=\frac{1}{2i}\,[(x+i)^{n+1}-(x-i)^{n+1}].$$

It is easy to compute that the roots of P_n are $\cot\frac{k\pi}{n+1}$, $k=1, 2, \ldots, n$.

712. Using the Euler formula to expand the identity

$$\frac{d^n\frac{x^2+1}{\sqrt{x^2+1}}}{dx^n}=\frac{d^{n-1}\frac{x}{\sqrt{x^2+1}}}{dx^{n-1}}$$

we get

$$P_n-(2n-1)\,xP_{n-1}+(n-1)^2\,(x^2+1)\,P_{n-2}=0.$$

Differentiating the equation defining P_{n-1}, we get

$$P_n-(2n-1)\,xP_{n-1}+(x^2+1)\,P'_{n-1}=0$$

whence

$$P'_{n-1}=(n-1)^2\,P_{n-2} \text{ and } P'_n=n^2\,P_{n-1}.$$

From the relations found, it follows that $P_n, P_{n-1}, \ldots, P_0=1$ form a Sturm sequence.

Since the leading coefficients are positive, all the roots of P_n are real.

713. The functions $F(x)$, $F'(x)$ and $[f'(x)]^2$ form a Sturm sequence for F. The leading coefficients of the sequence, $3a_0^2$, $12a_0^2$ and $9a_0^2$, are positive. Hence, the number of lost changes of sign when x passes from $-\infty$ to $+\infty$ is equal to two.

If f has a double root, then F has one triple root and one simple root. If f has a triple root, then F has a quadruple root.

714. If some one of the polynomials in the Sturm sequence has a multiple root x_0 or a complex root α, then this polynomial can be replaced by a polynomial of lower degree by dividing it by the positive quantity $(x-x_0)^2$ or $(x-\alpha)(x-\alpha')$. Subsequent polynomials can be replaced by remainders (taken with reversed signs) in the Euclidean algorithm for the replaced polynomial and the preceding one. Then the number of variations of sign for $x=-\infty$ will be $\leq n-2$, where n is the degree of the polynomial. Hence, the number of real roots is surely $\leq n-2$.

715. Let $F(x)=(x^2-1)^n$. $F(x)$ has -1 and $+1$ as roots of multiplicity n. $F'(x)$ has -1 and $+1$ as roots of multiplicity $n-1$ and, by Rolle's theorem, one more root in the interval $(-1, +1)$, $F''(x)$ has -1 and $+1$ as roots of multiplicity $n-2$ and two roots in the open interval $(-1, +1)$ and so on. $F^{(n)}(x)=P_n(x)$ has n roots in the open interval $(-1, 1)$.

716. Let $x_1, \dots, x_k$ be distinct roots of $f(x)$ of multiplicity $\alpha_1, \alpha_2, \dots, \alpha_k$, $x_1<x_2<x_3< \dots <x_k$. The function $\varphi(x)=\dfrac{f'(x)}{f(x)}$ is continuous in the open intervals $(-\infty, x_1)$, (x_1, x_2), $\dots$, (x_{k-1}, x_k) and $(x_k, +\infty)$ and ranges from 0 to $-\infty$ in the interval $(-\infty, x_1)$, from $+\infty$ to $-\infty$ in each of the intervals (x_{i-1}, x_i) and from $+\infty$ to 0 in the interval (x_k, ∞) because $\varphi(x)\to\infty$ as $x\to x_i$ and changes sign from $-$ to $+$ when passing through x_i.

Consequently, $\varphi(x)+\lambda$ has a root in each of the intervals (x_{i-1}, x_i) and, besides, for $\lambda>0$ one root in the interval $(-\infty, x_1)$, and for $\lambda<0$, one root in the interval $(x_k, +\infty)$.

Thus, $\varphi(x)+\lambda$ and, hence, $f(x)[\varphi(x)+\lambda]=\lambda f(x)+f'(x)$ also has k roots different from $x_1, x_2, \dots, x_k$ for $\lambda\neq 0$ or $k-1$ roots different from $x_1, x_2, \dots, x_k$ for $\lambda=0$. Besides, $\lambda f(x)+f'(x)$ has $x_1, x_2, \dots, x_k$ as roots of multiplicity α_1-1, $\alpha_2-1, \dots, \alpha_k-1$. Thus, the total number of real roots (counting multiplicity) of the polynomial $\lambda f(x)+f'(x)$ is equal to $\alpha_1+\alpha_2+\dots+\alpha_k$ for $\lambda\neq 0$ and $\alpha_1+\alpha_2+\dots+\alpha_k-1$ for $\lambda=0$, that is, it is equal to the degree of the polynomial $\lambda f(x)+f'(x)$.

717. Let $g(x)=a_0(x+\lambda_1)(x+\lambda_2)\dots(x+\lambda_n)$, $F_0(x)=a_0 f(x)$, $F_1(x)=F_0(x)+\lambda_1 F_0'(x)=a_0 f(x)+a_0\lambda_1 f'(x)$, $F_2(x)=F_1(x)+\lambda_2 F_1(x)=a_0 f(x)+a_0(\lambda_1+\lambda_2)f'(x)+a_0\lambda_1\lambda_2 f''(x)$, etc. Then $F_n(x)=F_{n-1}(x)+\lambda_n F_{n-1}'(x)=a_0 f(x)+a_1 f'(x)+\dots+a_n f^{(n)}(x)$ where $a_0, a_1, \dots, a_n$ are coefficients of g. By virtue of Problem 715, all roots of all polynomials $F_0, F_1, \dots, F_n$ are real.

718. The polynomial $a_0x^n+a_1mx^{n-1}+\dots+m(m-1)\dots(m-n+1)a_n=[a_0x^m+a_1(x^m)'+\dots+a_n(x^m)^{n}]x^{n-m}$ and all roots x^m are real.

719. The polynomial $a_nx^n+na_{n-1}x^{n-1}+n(n-1)a_{n-2}x^{n-2}+\dots+a_0n!$ has only real roots. Hence, all roots of $a_0n!x^n+a_1n(n-1)\dots 2x^{n-1}+\dots+na_{n-1}x+a_n$ are real. Applying once again the result of Problem 718, we find that all roots of the polynomial $a_0n!x^n+a_1n\cdot n(n-1)\dots 2x^{n-1}+a_2n(n-1)\cdot n(n-1)\dots 3x^{n-2}+\dots+a_nn!$ are real. It remains to divide by $n!$.

720. All roots of the polynomial $(1+x)^n=1+\frac{n}{1}x+\frac{n(n-1)}{1\cdot 2}x^2+\dots+x'$ are real. It remains to use the result of Problem 719.

721. The polynomial $f(x)=nx^n-x^{n-1}-x^{n-2}-\dots-1$ has a real root of 1. Furthermore, let $F(x)=(x-1)f(x)=nx^{n+1}-(n+1)x^n+1$. Then $F'(x)=n(n+1)(x-1)x^{n-1}$. For odd n, the polynomial $F(x)$ has a unique minimum for $x=1$ and, consequently, has no roots except the double root $x=1$. For even n, the polynomial $F(x)$ increases from $-\infty$ to 1 for $-\infty<x\leq 0$, decreases from 1 to 0 for $0\leq x\leq 1$ and increases from 0 to ∞ for $1\leq x<\infty$. Therefore, $F(x)$ in this case has a unique root other than the root $x=1$.

722. The derivative of the polynomial that interests us is positive for all real values of x. Hence, the polynomial has only one real root.

723. Let $a<b<c$; $f(-\infty)<0$; $f(a)=B^2(b-a)+C^2(c-a)>0$; $f(c)=-A^2(c-a)-B^2(c-b)<0$; $f(+\infty)>0$. Consequently, f has real roots in the intervals $(-\infty, a)$; (a, c); $(c, +\infty)$.

724. $\varphi(a+bi)$

$$=B+\sum_{k=1}^{n} \frac{A_k^2}{a+bi-a_k}=B+\sum_{k=1}^{n} \frac{A_k^2(a-a_k-bi)}{(a-a_k)^2+b^2},$$

$\operatorname{Im}\left(\varphi(a+bi)\right)=-b\sum_{k=1}^{n} \frac{A_k^2}{(a-a_k)^2+b^2}\neq 0$ for $b\neq 0$ because all terms under the summation sign are positive. Hence, $\varphi(a+bi)\neq 0$ for $b\neq 0$. The same result may also be obtained from the fact that $\varphi(x)$ varies from $+\infty$ to $-\infty$ as x varies from a_i to a_{i+1}, $\varphi(x)$ ranges from 0 to $-\infty$ for $-\infty<x<a_1$, $\varphi(x)$ varies from $+\infty$ to 0 for $a_n<x<\infty$. It is assumed here that

$$a_1<a_2<\ldots<a_n.$$

725. $\dfrac{f'(x)}{f(x)}=\sum_{k=1}^{n} \dfrac{1}{x-x_k}$, where x_k are roots of the polynomial $f(x)$.

Hence

$$[f'(x)]^2-f(x)f''(x)=[f(x)]^2\sum_{k=1}^{n} \frac{1}{(x-x_k)^2}>0$$

for all real values of x.

726. Let $x_1<x_2<\ldots<x_n$ be roots of the polynomial $f(x)$, and let $y_1<y_2<\ldots<y_m$ be roots of the polynomial $\varphi(x)$.

When the condition of the problem is fulfilled, $m=n$, $n-1$ or $n+1$. Without loss of generality, we can take it that $x_1<y_1<x_2<y_2<\ldots<y_{n-1}<x_n$ or $x_1<y_1<x_2<y_2<\ldots<y_{n-1}<x_n<y_n$. We assume $\lambda\neq 0$. Rewrite the equation as

$$\psi(x)=\frac{f(x)}{\varphi(x)}=-\frac{\mu}{\lambda}.$$

If $m=n$, then $\psi(x)$ varies:

from $\dfrac{a_0}{b_0}$ to $-\infty$ for $-\infty<x<y_1$, vanishing for $x=x_1$;

from $+\infty$ to $-\infty$ for $y_k<x<y_{k+1}$, vanishing for $x=x_{k+1}$;

from $+\infty$ to $\dfrac{a_0}{b_0}$ for $y_n<x<+\infty$. Here, a_0 and b_0 are the leading coefficients of $f(x)$ and $\varphi(x)$, which we consider to be positive.

Due to the continuity of $\psi(x)$ in each of the intervals under consideration, the equation $\psi(x)=-\dfrac{\mu}{\lambda}$ has n real roots if $-\dfrac{\mu}{\lambda}\neq\dfrac{a_0}{b_0}$ and $n-1$ real roots if $-\dfrac{\mu}{\lambda}=\dfrac{a_0}{b_0}$. Thus, the number of real roots of the equation $\lambda f(x)+\mu\varphi(x)$ is equal to its degree.

The case of $m=n-1$ is regarded in similar fashion.

727. The roots of $f(x)$ and $\varphi(x)$ are necessarily all real since $f(x)$ and $\varphi(x)$ are obtained from $F(x)$ for $\lambda=1$, $\mu=0$ and for $\mu=1$, $\lambda=0$.

Suppose the roots of $f(x)$ and $\varphi(x)$ are not separable. Without loss of generality, we can take it that there are no roots of $\varphi(x)$ between two adjacent roots x_1 and x_2 of the polynomial $f(x)$. Then $\psi(x)=\dfrac{f(x)}{\varphi(x)}$ is continuous for $x_1\leqslant x\leqslant x_2$ and vanishes at the endpoints of this interval. By Rolle's theorem, there is a point x_0 inside (x_1, x_2) such that $\psi'(x_0)=0$. Then $\psi(x)-\psi(x_0)$ has x_0 as a root of multiplicity $k\geqslant 2$. On the basis of the result of Problem 581, there are, on the circle $|z-x_0|=\rho$, if ρ is sufficiently small, at least four points at which $\operatorname{Im}\big(\psi(z)\big)=\operatorname{Im}\big(\psi(x_0)\big)=0$.

At least one of these points, z_0, is nonreal. The number $\mu=\psi(z_0)$ is real. The polynomial $F(x)=-f(x)+\mu\varphi(x)$ has a nonreal root, a contradiction.

728. The roots $\xi_1<\xi_2<\ldots<\xi_{n-1}$ of the polynomial $f'(x)$ divide the real axis into n intervals:

$$(-\infty, \xi_1), (\xi_1, \xi_2), \ldots, (\xi_{n-2}, \xi_{n-1}), (\xi_{n-1}, \infty).$$

By virtue of Rolle's theorem, in each of these intervals the polynomial $f(x)$ has at most one root. Furthermore, the polynomial $f'(x)+\lambda f''(x)$ for any real λ has at most one root in each of the foregoing intervals. Hence, $f(x)+\lambda f'(x)$, by virtue of Rolle's theorem, has at most two roots (counting multiplicity) in each of the intervals.

Separate all intervals into two classes. In the first put those which contain a root of $f(x)$. In the second, those without any root of $f(x)$. Consider the function $\psi(x)=\dfrac{f(x)}{f'(x)}$. In the intervals of Class One, $\psi(x)$ has one simple root and therefore changes sign. In the intervals of Class Two, $\psi(x)$ does not change sign. In the intervals of Class One, $\psi(x)+\lambda$ has an odd number of roots (counting multiplicity). Hence, from the foregoing, $\psi(x)+\lambda$ has only one simple root and no multiple roots in the interval of Class One. Therefore, $\psi'(x)$ has no roots in the interval of Class One. Now consider the intervals of Class Two. Let ξ_0 be a point in some interval of Class Two in which the absolute value of $\psi(x)$ reaches a minimum, and let $\lambda_0=\psi(\xi_0)$. For definiteness, we assume that $\psi(x)$ is positive in this interval. Then the function $\psi(x)-\lambda$ has no roots in the interval of our interest when $\lambda<\lambda_0$ and has at least two roots when $\lambda>\lambda_0$.

By virtue of what has been said, the number of roots of $\psi(x)-\lambda$ is exactly equal to two for $\lambda>\lambda_0$ and both roots are simple. Furthermore, $\psi(x)-\lambda_0$ has ξ_0 as a multiple (double) root.

Thus, $\psi(x)-\lambda$ has no multiple roots in the intervals of Class One and has only one multiple root for one value of λ in each interval of Class Two. Furthermore, each root η of the polynomial $f'^2(x)-f(x)f''(x)$ is a multiple root for $\psi(x)-\psi(\eta)$ since

$$[\psi(x)]'=\frac{f'^2(x)-f(x)f''(x)}{[f'(x)]^2}.$$

Thus, the number of real roots of $f'^2(x)-f(x)f''(x)$ is equal to the number of intervals of Class Two, which is obviously equal to the number of imaginary roots of $f(x)$.

729. $\lambda f_1(x)+\mu f_2(x)$ has all real roots for arbitrary real constants λ and μ (Problem 726). Hence, by virtue of Rolle's theorem, $\lambda f_1'(x)+\mu f_2'(x)$ has all real roots. From this it follows (Problem 727) that the roots of $f_1'(x)$ and $f_2'(x)$ are separable.

730. Suppose $f(x)$ has no multiple roots and let $\xi_1<\xi_2<\ldots<\xi_{n-1}$ be the roots of $f'(x)$. Consider the function $\psi(x)=\frac{f(x)}{f'(x)}+\frac{x+\lambda}{\gamma}$. It is obvious that $\lim\limits_{x\to\infty}\frac{\psi(x)}{x}=\frac{1}{n}+\frac{1}{\gamma}>0$ if $\gamma>0$ or if $\gamma<-n$. Whence it follows that $\psi(x)\to+\infty$ as $x\to+\infty$ and $\psi(x)\to-\infty$ as $x\to-\infty$. Besides $\psi(x)\to\to-\infty$ as $x\to\xi_i$ from the right and $\psi(x)\to+\infty$ as $x\to\xi_i$ from the left. Thus, $\psi(x)$ varies from $-\infty$ to $+\infty$ in each of the intervals $(-\infty, \xi_1)$, (ξ_1, ξ_2), $\ldots$, (ξ_{n-1}, ∞) remaining continuous inside these intervals.

Consequently, $\psi(x)$, and its numerator $\gamma f(x)+(x+\lambda)f'(x)$ as well, has at least n distinct roots for $\gamma>0$ or $\gamma<-n$. But the number of roots of $\gamma f(x)+ +(x+\lambda)f'(x)$ does not exceed n, because $\gamma f(x)+(x+\lambda)f'(x)$ is a polynomial of degree n. If $f(x)$ has multiple roots and $x_1, x_2, \ldots, x_k$ are distinct roots of $f(x)$, then $f'(x)$ has $k-1$ roots $\xi_1, \xi_2, \ldots, \xi_{k-1}$ different from $x_1, x_2, \ldots, x_k$. Reasoning in like fashion, we are satisfied of the existence of k roots of $\psi(x)$. All of them, except $-\lambda$ if $-\lambda$ is among the roots of $f(x)$, will be different from the roots of $f(x)$.

Besides these roots, $\gamma f(x)+(x+\lambda)f'(x)$ will have as roots $x_1, x_2, \ldots, x_k$ with the sum of multiplicities $n-k$ [if $-\lambda$ is not a root of $f(x)$] or $n-k+1$ [if $-\lambda$ is a root of $f(x)$].

The total number of real roots of $\gamma f(x)+(\lambda+x)f'(x)$ counting multiplicities is again equal to n.

731. Let $\varphi(x)=b_k(x+\gamma_1)(x+\gamma_2)\ldots(x+\gamma_k)$. Every γ_i is either greater than zero or less than $-n$.

It is obvious that the coefficients of the polynomial

$$F_1(x)=\gamma_1 f(x)+xf'(x)$$

are $a_i(\gamma_1+i)$. The coefficients of the polynomial

$$F_2(x)=\gamma_2 F_1(x)+x F_1'(x)$$

are $a_i(\gamma_1+i)(\gamma_2+i)$ and so on, the coefficients of the polynomial

$$F_k(x)=\gamma_k F_{k-1}(x)+xF_{k-1}'(x)$$

are $a_i(\gamma_1+i)(\gamma_2+i)\ldots(\gamma_k+i)$, $i=1, 2, \ldots, n$.

On the basis of the result of Problem 730, all roots of all polynomials F_1, $F_2, \ldots, F_k$ are real. But

$$a_0\varphi(0)+a_1\varphi(1)x+\ldots+a_n\varphi(n)x^n=b_k F_k(x).$$

732. Suppose $f(x)=f_1(x)(x+\lambda)$ where λ is a real number and $f_1(x)$ is a polynomial of degree $n-1$, all the roots of which are real. Suppose that for polynomials of degree $n-1$ the theorem is valid; on this assumption, prove it for polynomials of degree n.

$$\text{Let } f_1(x)=b_0+b_1x+\ldots+b_{n-1}x^{n-1},$$
$$f(x)=a_0+a_1x+\ldots+a_nx^n.$$

Then

$$\begin{aligned}
a_0 &= \lambda b_0,\\
a_1 &= \lambda b_1+b_0,\\
a_2 &= \lambda b_2+b_1,\\
&\ldots\ldots\ldots\ldots\\
a_{n-1} &= \lambda b_{n-1}+b_{n-2},\\
a_n &= b_{n-1}
\end{aligned}$$

and

$$a_0+a_1\gamma x+a_2\gamma(\gamma-1)x^2+\dots+a_n\gamma(\gamma-1)\dots(\gamma-n+1)x^n=\lambda[b_0+b_1\gamma x+b_2\gamma(\gamma-1)x^2+\dots+b_{n-1}\gamma(\gamma-1)\dots(\gamma-n+2)x^{n-1}]+x\gamma[b_0+b_1(\gamma-1)x+b_2(\gamma-1)(\gamma-2)x^2+\dots+b_{n-1}(\gamma-1)(\gamma-2)\dots(\gamma-n+1)x^{n-1}]=\lambda\varphi(x)+x[\gamma\varphi(x)-x\varphi'(x)]$$

where $\varphi(x)$ is used to denote the polynomial

$$b_0+b_1\gamma x+b_2\gamma(\gamma-1)x^2+\dots+b_{n-1}\gamma(\gamma-1)\dots(\gamma-n+2)x^{n-1}.$$

By hypothesis, all roots of the polynomial $\varphi(x)$ are real. It remains to prove the following lemma.

Lemma. If $\varphi(x)$ is a polynomial of degree $n-1$ having only real roots, then all roots of the polynomial $\psi(x)=\lambda\varphi+\gamma x\varphi-x^2\varphi'$ are real for $\gamma>n-1$ and for arbitrary real λ.

Proof. Without loss of generality, we can take it that 0 is not a root of $\varphi(x)$ for if $\varphi=x^k\varphi_1$, $\varphi_1(0)\neq 0$, then

$$\psi(x)=x^k(\lambda\varphi_1+(\gamma-k)x\varphi_1-x^2\varphi_1')=x^k\psi_1$$

and $\gamma_1=\gamma-k$ exceeds the degree of φ_1.

Let $x_1, x_2, \dots, x_m$ be distinct roots of φ. The polynomial ψ has among its roots $x_1, x_2, \dots, x_m$ with sum of multiplicities $n-1-m$. Now consider

$$w(x)=\lambda+\gamma x-\frac{x^2\varphi'(x)}{\varphi(x)}.$$

It is obvious that

$$\lim_{x\to\infty}\frac{w(x)}{x}=\gamma-(n-1)>0.$$

Hence, $w(x)\to-\infty$ as $x\to-\infty$ and $w(x)\to+\infty$ as $x\to+\infty$. Besides, $w(x)\to\to+\infty$ as $x\to x_i$ from the left and $w(x)\to-\infty$ as $x\to x_i$ from the right. For this reason $w(x)$ has roots in each of the intervals

$$(-\infty, x_1), (x_1, x_2), \dots, (x_{m-1}, x_m), (x_m, +\infty).$$

The total number of real roots of $\psi(x)$, counting multiplicity, is equal to $n-1-m+m+1=n$, that is, it is equal to the degree of $\psi(x)$, which is what we set out to prove.

733. If all the roots of the polynomial $a_0+a_1x+\dots+a_nx^n$ are real, then all the roots of the polynomial $a_0x^n+a_1x^{n-1}+\dots+a_n$ are real. Furthermore, all the roots of the polynomials

$$a_0\gamma_1(\gamma_1-1)\dots(\gamma_1-n+1)x^n+a_1\gamma_1(\gamma_1-1)\dots(\gamma_1-n+2)x^{n-1}+\dots+a_{n-1}\gamma_1x+a_n$$

and

$$a_0\gamma_1(\gamma_1-1)\dots(\gamma_1-n+1)+a_1\gamma_1(\gamma_1-1)\dots(\gamma_1-n+2)x+\dots+a_{n-1}\gamma_1x^{n-1}+a_nx^n$$
$$=\left[a_0+\frac{a_1}{\gamma_1-n+1}x+\dots+\frac{a_{n-1}}{(\gamma_1-n+1)(\gamma_1-n+2)\dots(\gamma_1-1)}x^{n-1}+\frac{a_n}{(\gamma_1-n+1)(\gamma_1-n+2)\dots\gamma_1}x^n\right]\gamma_1(\gamma_1+1)\dots(\gamma_1-n+1)$$

are real for $\gamma_1 > n-1$. Setting $\gamma_1 - n + 1 = \alpha > 0$, we find that all the roots of the polynomial

$$a_0 + \frac{a_1}{\alpha}\, x + \frac{a_2}{\alpha(\alpha+1)}\, x^2 + \ldots + \frac{a_n}{\alpha(\alpha+1)\ \ldots\ (\alpha+n-1)}\, x^n$$

are real. Using the result of Problem 732 a second time, we get the desired result.

734. 1. Suppose all the roots of $f(x)$ are positive. Then the polynomial $a_0 + a_1 wx + \ldots + a_n w^{n^2} x^n$ cannot have negative roots. Suppose the theorem holds true for polynomials of degree $n-1$. Denote

$$\varphi(x) = b_0 + b_1 wx + b_2 w^4 x^2 + \ldots + b_{n-1} w^{(n-1)^2} x^{n-1}.$$

Let $0 < x_1 < x_2 < \ldots < x_{n-1}$ where $x_1, x_2, \ldots, x_{n-1}$ are roots of $\varphi(x)$ and let $\frac{x_i}{x_{i-1}} > w^{-2}$.

Further suppose that $f(x) = (\lambda - x)(b_0 + b_1 x + \ldots + b_{n-1} x^{n-1})$. The coefficients of the polynomial $f(x)$ are

$$\begin{aligned}
a_0 &= \lambda b_0,\\
a_1 &= \lambda b_1 - b_0,\\
a_2 &= \lambda b_2 - b_1,\\
&\ldots\ldots\ldots\ldots\\
a_{n-1} &= \lambda b_{n-1} - b_{n-2},\\
a_n &= -b_{n-1}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\psi(x) &= a_0 + a_1 wx + a_2 w^4 x^2 + \ldots + a_n w^{n^2} x^n = \lambda(b_0 + b_1 wx\\
&\quad + \ldots + b_{n-1} w^{(n-1)^2} x^{n-1}) - x(b_0 w + b_1 w^4 x + \ldots + b_{n-1} w^{n^2} x^{n-1})\\
&= \lambda\varphi(x) - xw\varphi(xw^2).
\end{aligned}$$

The roots of the polynomials $\varphi(x)$ and $x\varphi(xw^2)$ are separated by virtue of the induction hypothesis. Thus, all the roots of the polynomial $\lambda\varphi(x) + xw\varphi(xw^2)$ that interests us are real. It remains to verify that the law of their distribution is the same as for the polynomial $\varphi(x)$.

Denote by $z_1, z_2, \ldots, z_n$ the roots of $\psi(x)$. It is easy to see that

$$0 < z_1 < x_1 < x_1 w^{-2} < z_2 < x_2 < x_2 w^{-2} < z_3 < \ldots < z_{n-1} < x_{n-1} < x_{n-1} w^{-2} < z_n.$$

Whence it follows that $\frac{z_i}{z_{i-1}} > \frac{x_{i-1} w^{-2}}{x_{i-1}} = w^{-2}$, which completes the proof.

2. Consider $\varphi_m(x) = \left(1 - \dfrac{x^2 \log \frac{1}{w}}{m}\right)^m$.

For m sufficiently large, the roots of the polynomial $\varphi_m(x)$, equal to $\pm\sqrt{\dfrac{m}{\log \frac{1}{w}}}$, do not lie in the interval $(0, n)$. Consequently (Problem 731), all the roots of the polynomial $a_0 \varphi_m(0) + a_1 \varphi_m(1) x + \ldots + a_n \varphi_m(n) x^n$ are real. But $\lim_{m\to\infty} \varphi_m(x) = w^{x^2}$. Hence, by virtue of the continuity of the roots as functions of the coefficients, all the roots of $a_0 + a_1 wx + \ldots + a_n w^{n^2} x^n$ are real.

735. Denote by $x_1, x_2, \ldots, x_n$ the roots of the polynomial $f(x)=a_0+a_1x+\ldots+a_nx^n$. Without loss of generality, they may be considered positive. Furthermore, let

$$\varphi(x)=a_0\cos\varphi+a_1\cos(\varphi+\theta)x+\ldots+a_n\cos(\varphi+n\theta)x^n,$$

$$\psi(x)=a_0\sin\varphi+a_1\sin(\varphi+\theta)x+\ldots+a_n\sin(\varphi+n\theta)x^n.$$

Then

$$\varphi(x)+i\psi(x)=(\cos\varphi+i\sin\varphi)a_n\prod_{i=1}^{n}(\alpha x-x_i),$$

$$\varphi(x)-i\psi(x)=(\cos\varphi-i\sin\varphi)a_n\prod_{i=1}^{n}(\alpha' x-x_i)$$

where

$$\alpha=\cos\theta+i\sin\theta,\ \alpha'=\cos\theta-i\sin\theta.$$

Consequently,

$$\Phi(x)=\frac{\varphi(x)+i\psi(x)}{\varphi(x)-i\psi(x)}=\frac{\cos\varphi+i\sin\varphi}{\cos\varphi-i\sin\varphi}\prod_{i=1}^{n}\frac{\alpha x-x_i}{\alpha' x-x_i}.$$

Let $x=\rho\beta$ be a root of the polynomial $\varphi(x)$. Here, $\rho=|x|$; $\beta=\cos\lambda+i\sin\lambda$. Then $|\Phi(x)|=1$ and, hence,

$$\prod_{i=1}^{n}\left|\frac{\rho\alpha\beta-x_i}{\rho\alpha'\beta-x_i}\right|=1$$

but

$$\left|\frac{\rho\alpha\beta-x_i}{\rho\alpha'\beta-x_i}\right|^2=\frac{(\rho\alpha\beta-x_i)(\rho\alpha'\beta'-x_i)}{(\rho\alpha'\beta-x_i)(\rho\alpha\beta'-x_i)}$$

$$=1+\frac{\rho x_i(\alpha-\alpha')(\beta'-\beta)}{|\rho\alpha'\beta-x_i|^2}=1+\frac{4\rho x_i\sin\Theta\sin\lambda}{|\rho\alpha'\beta-x_i|^2}$$

We disregard the uninteresting case of $\sin\theta=0$.

If $\sin\lambda\neq0$, then all $\left|\frac{\rho\alpha\beta-x_i}{\rho\alpha'\beta-x_i}\right|^2$ are simultaneously greater than unity or simultaneously less than unity and their product cannot be equal to 1. Hence, $\sin\lambda=0$, which means x is real.

736. Let $x_1, x_2, \ldots, x_n$ be the roots of the polynomial

$$f(x)=a_0+ib_0+(a_1+ib_1)x+\ldots+(a_n+ib_n)\lambda^n=\varphi(x)+i\psi(x).$$

The imaginary parts of these roots are positive. Let us consider the polynomial $\tilde{f}(x)=\varphi(x)-i\psi(x)$. Its roots will obviously be $x'_1, x'_2, \ldots, x'_n$, which are conjugate to $x_1, x_2, \ldots, x_n$. Then

$$\Phi(x)=\frac{\varphi(x)+i\psi(x)}{\varphi(x)-i\psi(x)}=\prod_{i=1}^{n}\frac{x-x_i}{x-x'_i}\cdot\frac{a_n+ib_n}{a_n-ib_n}.$$

If x_0 is a root of $\varphi(x)$, then

$$|\Phi(x_0)| = \prod_{i=1}^{n} \left| \frac{x_0 - x_i}{x_0 - x_i'} \right| = 1.$$

But

$$\left| \frac{x_0 - x_i}{x_0 - x_i'} \right|^2 = \frac{(x_0 - x_i)(x_0' - x_i')}{(x_0 - x_i')(x_0' - x_i)} = 1 + \frac{(x_i - x_i')(x_0 - x_0')}{|x_0 - x_i|^2}$$

$$= 1 - \frac{4 \operatorname{Im}(x_0) \operatorname{Im}(x_i)}{|x_0 - x_i'|^2}.$$

Whence, if $\operatorname{Im}(x_0) > 0$, then $\left| \frac{x_0 - x_i}{x_0 - x_i'} \right| < 1$ for all i; if $\operatorname{Im}(x_0) < 0$, then $\left| \frac{x_0 - x_i}{x_0 - x_i'} \right| > 1$ for all i. (The same thing can be obtained geometrically with ease and without computations.) Thus, the equation $|\Phi(x_0)| = 1$ is only possible for a real x_0 and therefore all roots of $\varphi(x)$ are real.

Next, consider the polynomial

$$(\alpha - \beta i)[\varphi(x) + i\psi(x)] = \alpha\varphi(x) + \beta\psi(x) + i[\alpha\psi(x) - \beta\varphi(x)].$$

Its roots do not differ from the roots of the original polynomial and, hence, its real part $\alpha\varphi(x) + \beta\psi(x)$ only has real roots for arbitrary real α and β. But in this case, the roots $\varphi(x)$ and $\psi(x)$ can be separated (Problem 727).

737. Let $x_1, x_2, \ldots, x_n$ be the roots of $\varphi(x)$; $y_1, y_2, \ldots, y_n$ the roots of $\psi(x)$. Without loss of generality, we can assume that the leading coefficients of φ and ψ are positive and

$$x_1 > y_1 > x_2 > y_2 > \ldots > y_{n-1} > x_n > y_n$$

(y_n may be absent).

Decompose $\frac{\psi(x)}{\varphi(x)}$ into partial fractions

$$\frac{\psi(x)}{\varphi(x)} = A + \sum_{k=1}^{n} \frac{A_k}{x - x_k}, \quad A_k = \frac{\psi(x_k)}{\varphi'(x_k)}.$$

It is easy to see that all $A_k > 0$. Set $x = a + bi$ and find the imaginary part of

$$\frac{-i(\varphi(x) + i\psi(x))}{\varphi(x)} = \frac{\psi(x)}{\varphi(x)} - i,$$

$$\operatorname{Im}\left(\frac{\psi(x)}{\varphi(x)} - i\right) = -1 + \operatorname{Im}\left(\sum_{k=1}^{n} \frac{A_k}{a + bi - x_k}\right)$$

$$= -1 - b \sum_{k=1}^{n} \frac{A_k}{(a - x_k)^2 + b^2}.$$

If $b \geqslant 0$, then $\operatorname{Im}\left(\frac{\psi(x)}{\varphi(x)}-i\right)<0$ and, hence, $\varphi(x)+i\psi(x)\neq 0$. Thus, in the case at hand, all the roots of $\varphi(x)+i\psi(x)$ lie in the lower half-plane. The other cases of location of roots are considered in similar fashion.

738.

$$\frac{f'(x)}{f(x)}=\sum_{k=1}^{n}\frac{1}{x-x_k}, \quad x_k \text{ are roots of } f(x).$$

Let $x=a-bi$, $b>0$. Then

$$\operatorname{Im}\left(\frac{f'(a-bi)}{f(a-bi)}\right)=\sum_{k=1}^{n}\frac{b+\operatorname{Im}(x_k)}{|x-x_k|^2}>0.$$

Consequently,

$$f'(a-bi)\neq 0.$$

739. Let the half-plane be represented by the inequality

$$r\cos(\theta-\varphi)>p, \text{ where } x=r(\cos\varphi+i\sin\varphi).$$

Put $x=(x'+pi)(\sin\theta-i\cos\theta)$. Then

$$x'=-pi+x(\sin\theta+i\cos\theta)=r\sin(\theta-\varphi)+i[r\cos(\theta-\varphi)-p].$$

Whence it follows that if x lies in the given half-plane, then x' lies in the half-plane $\operatorname{Im}(x')>0$, and conversely. The roots of the polynomial $f[(x'+pi)(\sin\theta-i\cos\theta)]$ are thus located in the upper half-plane. On the basis of Problem 738, the roots of its derivative, equal to $[\sin\theta-i\cos\theta]\, f'[(x'+pi)(\sin\theta-i\cos\theta)]$, also lie in the upper half-plane.

Thus, the roots of the polynomial $f'(x)$ lie in the given half-plane.

740. This follows immediately from the result of Problem 739.

741. The equation splits into two:

$$\frac{f'(x)}{f(x)}+\frac{1}{ki}=0 \text{ and } \frac{f'(x)}{f(x)}-\frac{1}{ki}=0.$$

Decomposition into partial fractions yields

$$\sum_{k=1}^{n}\frac{1}{x-x_k}\pm\frac{1}{ki}=0$$

x_k are the roots of $f(x)$, which, by hypothesis, are real. Let $x=a+bi$. Then

$$\left|\operatorname{Im}\left(\sum_{k=1}^{n}\frac{1}{x-x_k}\right)\right|=|b|\sum_{k=1}^{n}\frac{1}{(a-x_k)^2+b^2}<\frac{n}{|b|}.$$

For the roots of each of the equations it must be true that $\frac{1}{k}<\frac{n}{|b|}$, whence $|b|<kn$.

742. All the roots of $f'(x)$ are obviously real. Denote them by $\xi_1, \xi_2, \ldots, \xi_{n-1}$. Next, denote by $y_1, y_2, \ldots, y_n$ the roots of the polynomial $f(x)-b$, by $x_1, x_2, \ldots, x_n$ the roots of the polynomial $f(x)-a$. Then

$$y_1<\xi_1<y_2<\xi_2< \ldots <y_{n-1}<\xi_{n-1}<y_n,$$

$$x_1<\xi_1 <x_2<\xi_2< \ldots <x_{n-1}<\xi_{n-1}<x_n.$$

From these inequalities it follows that intervals bounded by the points x_i, y_i do not overlap since they lie in the nonoverlapping intervals

$$(-\infty, \xi_1), (\xi_1, \xi_2), \ldots, (\xi_{n-1}, +\infty).$$

The polynomial $f(x)$ takes on the values a and b at the endpoints of each of these intervals and passes through all intermediate values inside the interval. Consequently, $f(x)-\lambda$ vanishes n times on the real axis, which completes the proof.

743. If the real parts of the roots of the polynomial $f(x)=x^n+a_1x^{n-1}+ + \ldots +a_n$ have like signs, then the imaginary parts of the roots of the polynomial

$$i^n f(-ix)=x^n+ia_1x^{n-1}-a_2x^{n-2}-ia_3x^{n-3}+ \ldots$$

also have like signs, and conversely.

For this, by virtue of the result of Problems 736, 737, it is necessary and sufficient that the roots of the polynomials $x^n-a_2x^{n-2}+a_4x^{n-4}- \ldots$ and $a_1x^{n-1}-a_3x^{n-3}+a_5x^{n-5}- \ldots$ be real and separable.

744. It is necessary that $a>0$ and that the roots of the polynomials x^3- $-bx$ and ax^2-c be real and separable. For this, the necessary and sufficient condition is $0<\frac{c}{a}<b$ or $c>0$, $ab-c>0$.

Thus, for negativity of the real parts of all roots of the equation

$$x^3+ax^2+bx+c=0$$

it is necessary and sufficient that the inequalities $a>0$, $c>0$, $ab-c>0$ be fulfilled.

745. $a>0$, $c>0$, $d>0$, $abc-c^2-a^2d>0$.

746. Set $x=\frac{1+y}{1-y}$. It is easy to see that if $|x|<1$, then the real part of y is negative, and conversely.

Consequently, for all roots x_1, x_2, x_3 of the equation $f(x)=0$ to be less than 1 in absolute value, it is necessary and sufficient that all the roots of the equation $f\left(\frac{1+y}{1-y}\right)=0$ have negative real parts. This equation is of the form

$$y^3(1-a+b-c)+y^2(3-a-b+3c)+y(3+a-b-3c)+(1+a+b+c)=0.$$

Besides, it is easy to see the necessity of the condition

$$1-a+b-c=(1+x_1)(1+x_2)(1+x_3)>0.$$

On the basis of the results of Problem 744, we get the necessary and sufficient conditions:

$$1-a+b-c>0,\ 1+a+b+c>0,\ 3-a-b+3c>0,\ 1-b+ac-c^2>0.$$

747. $f(x)\,(1-x)=a_n+(a_{n-1}-a_n)\,x$

$$+(a_{n-2}-a_{n-1})\,x^2+\ \dots\ +(a_0-a_1)x^n-a_0x^{n+1}.$$

Let $|x|=\rho>1$. Then

$$|f(x)\ (1-x)|\geqslant a_0\rho^{n+1}-|a_n+(a_{n-1}-a_n)\,x$$

$$+\ \dots\ +(a_0-a_1)\,x^n|\geqslant a_0\,\rho^{n+1}-\rho^n\,(a_n+a_{n-1}-a_n$$

$$+\ \dots\ +a_0-a_1)=a_0\ (\rho^{n+1}-\rho^n)>0.$$

Consequently, $f(x)\neq 0$ for $|x|>1$.

748. -0.6618. **749.** 2. 094551.

750. (a) 3.3876, -0.5136, -2.8741; (b) 2.8931; (c) 3.9489, 0.2172, -1.1660; (d) 3.1149, 0.7459, -0.8608.

751. The problem reduces to computing the root of the equation $x^3-3x+1=0$ contained in the interval (0, 1).

Answer: $x=0.347$ (to within 0.001).

752. 2.4908.

753. (a) 1.7320, (b) -0.7321, (c) 0.6180, (d) 0.2679,
(e) $-3.$ 1623, (f) 1.2361, (g) -2.3028, (h) 3.6457, (i) 1.6180.

754. (a) 1.0953, -0.2624, -1.4773, -2.3556; (b) 0.8270, 0.3383, -1.2090, -2.9563;
(c) 1.4689, 0.1168;
(d) 8.0060, 1.2855, 0.1960, -1.4875;
(e) 1.5357, -0.1537;
(f) 3.3322, 1.0947, -0.6002, -1.8268;
(g) 0.4910, -1.4910,
(h) 2.1462, -0.6821, -1.3178, -4.1463.

CHAPTER 6

SYMMETRIC FUNCTIONS

755. The following is a detailed solution of Example (f):

$$F(x_1,\ x_2,\ x_3)=(x_1^2+x_2^2)\,(x_1^2+x_3^2)\,(x_2^2+x_3^2)$$

The leading term of the polynomial F is $x_1^4\cdot x_2^2$.

Write out the exponents in the leading terms of the polynomials that will remain after a successive elimination of the leading terms due to subtracting appropriate combinations of the elementary symmetric polynomials. The exponents are:

$$(4,\ 2,\ 0),\ (4,\ 1,\ 1),\ (3,\ 3,\ 0),\ (3,\ 2,\ 1)\ \text{and}\ (2,\ 2,\ 2).$$

Hence, $F=f_1^2f_2^2+Af_1^3f_3+Bf_2^3+Cf_1f_2f_3+Df_3^2$ where A, B, C, D are numerical coefficients. We determine them, specifying particular values for x_1, x_2, x_3.

x_1	x_2	x_3	f_1	f_2	f_3	F
1	1	0	2	1	0	2
2	-1	-1	0	-3	2	50
1	-2	-2	-3	0	4	200
1	-1	-1	-1	-1	1	8

We have the following system of equations for determining A, B, C, D:

$$2=4+B,$$

$$50=-27B+4D,$$

$$200=-108A+16D,$$

$$8=1-A-B+C+D$$

whence $B=-2$, $D=-1$, $A=-2$, $C=4$.

Thus,

$$(x_1^2+x_2^2)(x_1^2+x_3^2)(x_2^2+x_3^2)=f_1^2f_2^2-2f_1^3f_3-2f_2^3+4f_1f_2f_3-f_3^2.$$

The answers for the other examples are:

(a) $f_1^3-3f_1f_2$; (b) $f_1f_2-3f_3$; (c) $f_1^4-4f_1^2f_2+8f_1f_3$;

(d) $f_1^3f_2^2-2f_1^4f_3-3f_1f_2^3+6f_1^2f_2f_3+3f_2^2f_3-7f_1f_3^2$;

(e) $f_1f_2-f_3$; (g) $2f_1^3-9f_1f_2+27f_3$;

(h) $f_1^2f_2^2-4f_1^3f_3-4f_2^3+18f_1f_2f_3-27f_3^2$.

756. (a) $f_1f_2f_3-f_1^2f_4-f_3^2$; (b) $f_1^2f_4+f_3^2-4f_2f_4$;

(c) $f^3-4f_1f_2+8f_3$.

757. (a) $f_1^2-2f_2$; (b) $f^3-3f_1f_2+3f_3$;

(c) $f_1f_3-4f_4$; (d) $f_2^2-2f_1f_3+2f_4$;

(e) $f_1^2f_2-f_1f_3-2f_2+4f_4$; (f) $f_1^4-4f_1^2f_2+2f_2^2+4f_1f_3-4f_4$;

(g) $f_2f_3-3f_1f_4+5f_5$; (h) $f_1^2f_3-2f_2f_3-f_1f_4+5f_5$;

(i) $f_1f_2^2-2f_1^2f_3-f_2f_3+5f_1f_4-5f_5$;

(j) $f_1^3f_2-3f_1f_2^2-f_1^2f_3+5f_2f_3+f_1f_4-5f_5$;

(k) $f_1^5-5f_1^3f_2+5f_1f_2^2+5f_1^2f_3-5f_2f_3-5f_1f_4+5f_5$;

(l) $f_2f_4-4f_1f_5+9f_6$; (m) $f_3^2-2f_2f_4+2f_1f_5-2f_6$;

(n) $f_1^2f_4-2f_2f_4-f_1f_5+6f_6$;

(o) $f_1f_2f_3-3f_1^2f_4-3f_3^2+4f_2f_4+7f_1f_5-12f_6$;

(p) $f_2^3-3f_1f_2f_3+3f_1^2f_4+3f_3^2-3f_2f_4-3f_1f_5+3f_6$;

(q) $f_1^3f_3-3f_1f_2f_3-f_1^2f_4+3f_3^2+2f_2f_4+f_1f_5-6f_6$;

(r) $f_1^2f_2^2-2f_1^3f_3-2f^3+4f_1f_2f_3+2f_1^2f_4-3f_3^2+2f_2f_4-6f_1f_5+6f_6$;

(s) $f_1^4f_2-4f_1^2f_2^2-f_1^3f_3+2f_2^3+7f_1f_2f_3+f_1^2f_4-3f_3^2-6f_2f_4-f_1f_5+6f_6$;

(t) $f_1^6-6f_1^4f_2+9f_1^2f_2^2+6f_1^3f_3-2f_2^3-12f_1f_2f_3-6f_1^2f_4$
$$+3f_3^2+6f_2f_4+6f_1f_5-6f_6.$$

758. (a) nf^2-8f_2;

(b) $-f_1^n \quad +4f_1^{n-2}f_2-8f_1^{n-3}f_3+\ldots+(-2)^nf_n.$

759. (a) $(n-1)f_1^2-2nf_2$; (b) $(n-1)f_1^3-3(n-2)f_1f_2+3(n-4)f_3$;

(c) $(n-1)f_1^4-4nf_1^2f_2+2(n+6)f_2^2+4(n-3)f_1f_3-4nf_4$;

(d) $\dfrac{3(n-1)(n-2)}{2}f_1^2-(3n-1)(n-2)f_2.$

760. $f_k^2-2f_{k-1}f_{k+1}+2f_{k-2}f_{k+2}-2f_{k-3}f_{k+3}+\ldots$

761. $$(n-1)!\sum_{i=1}^{n}a_i^2f^2-2(n-2)!\left[n\sum_{i=1}^{n}a_i^2-\left(\sum_{i=1}^{n}a_i\right)^2\right]f_2$$
$$=(n-1)!\,S_2s_2+4(n-2)!\,F_2f_2$$

where

$$S_2=\sum_{i=1}^{n}a_i^2;\ s_2=\sum_{i=1}^{n}x_i^2;\ F_2=\sum_{i<k}a_ia_k;\ f_2=\sum_{i<k}x_ix_k.$$

762. (a) $\dfrac{f_1f_2-3f_3}{f_3}$; (b) $\dfrac{2(f_1^2f_2-3f_1f_3-2f_2^2)}{f_1f_2-f_3}$;

(c) $\dfrac{f_2^3+f^3f_3-6f_1f_2f_3+9f_3^2}{f^2}$.

763. (a) $\dfrac{f_2^2-2f_1f_3+2f_4}{f_4}$; (b) $\dfrac{f_1^2f_2^2+f_1^3f_3-6f_1f_2f_3+6f_3^2+2f_1^2f_4}{f_1f_2f_3-f_1^2f_4-f_3^2}$.

764. (a) $\dfrac{f_{n-1}}{f_n}$; (b) $\dfrac{f_{n-1}^2-2f_{n-2}f_n}{f_n^2}$; (c) $\dfrac{f_1f_{n-1}-nf_n}{f_n}$;

(d) $\dfrac{f_1^2f_{n-1}^2-2f_2f_{n-1}^2-2f_1^2f_{n-2}f_n+4f_2f_{n-2}f_n-nf_n^2}{f_n^2}$;

(e) $\dfrac{f_1^2f_{n-1}-2f_2f_{n-1}-f_1f_n}{f_n}$; (f) $\dfrac{f_2f_{n-1}-(n-1)f_1f_n}{f_n}$.

765. -4. **766.** -35. **767.** 16.

768. (a) -3; (b) $-2p^3-3q^2$; (c) $-p^3\,(x_1^2-x_2x_3=-p)$;

(d) q^4; (e) $\dfrac{-2p-3q}{1+p-q}$; (f) $\dfrac{2p^2-4p-4pq+3q^2+6q}{(1+p-q)^2}$.

769. Let $x_1^2=x_2^2+x_3^2$. Then $2x_1^2=x_1^2+x_2^2+x_3^2=a^2-2b$. Hence, $\sqrt{\dfrac{a^2-2b}{2}}$ or $-\sqrt{\dfrac{a^2-2b}{2}}$ is among the roots of the given equation. For this, it is necessary and sufficient that the following condition be fulfilled:

$$a^4\,(a^2-2b)=2\,(a^3-2ab+2c)^2.$$

770. $a=-x_1-x_2-x_3$,

$$ab-c=-(x_1+x_2)\,(x_1+x_3)\,(x_2+x_3),$$

$$c=-x_1x_2x_3.$$

If all roots are real and negative, then

$$a>0,\ b>0,\ c>0.$$

If one root x_1 is real and x_2 and x_3 are complex conjugate roots with negative real part, then $x_2+x_3<0$, $x_2x_3>0$, $(x_1+x_2)\,(x_1+x_3)>0$ and, hence, also $a>0$, $b>0$ and $c>0$. The necessity of the conditions is proved.

Now assume that $a>0$, $b>0$, $c>0$. If x_1 is real and x_2 and x_3 are complex conjugates, then $x_2x_3>0$, $(x_1+x_2)\,(x_1+x_3)>0$ and from $c>0$, $b>0$ it follows that $x_1<0$, $2\text{Re}\,(x_2)=x_2+x_3<0$.

Now if x_1, x_2, x_3 are real, then from $c>0$ it follows that one root, x_1, is negative, and the other two are of the same sign. If $x_2>0$, $x_3>0$, it follows that

$$-x_1-x_2>x_3>0,\quad -x_1-x_3>x_2>0$$

and then $-(x_1+x_2)\,(x_1+x_3)\,(x_2+x_3)<0$, which contradicts the hypothesis. Hence, $x_2<0$, $x_3<0$.

An alternative solution is given in Problem 744.

771. $s=\dfrac{1}{4}\sqrt{a\,(4ab-a^3-bc)}$, $R=\dfrac{c}{\sqrt{a\,(4ab-a^3-bc)}}$.

772. $a\,(4ab-a^3-8c)=4c^2$.

773. (a) $\dfrac{25}{27}$, (b) $\dfrac{35}{27}$, (c) $-\dfrac{1{,}679}{625}$.

774. (a) $a_1^2a_2^2-4a_1^3a_3-4a_2^3a_0+18a_0a_1a_2a_3-27a_0^2a_3^2$;

(b) $a_1^3a_3-a_2^3a_0$; (c) $\dfrac{a_1a_2}{a_0a_3}-9$; (d) $a_1^2a_2^2-a_1^3a_3-a_2^3a_0$.

775. It is sufficient to give the proof for the elementary symmetric polynomials. Let φ_k be an elementary symmetric polynomial of $x_2, x_3, \ldots, x_n$ of degree k; let f_k be an elementary symmetric polynomial of $x_1, x_2, \ldots, x_n$. It is obvious that $\varphi_k = f_k - x_1 \varphi_{k-1}$, whence it follows that

$$\varphi_k = f_k - x_1 f_{k-1} + x_1^2 f_{k-2} - \ldots + (-x_1)^{k-1} f_1 + (-1)^k x_1^k,$$
$$(-1)^k \varphi_k = a_k + a_{k-1} x_1 + \ldots + a_1 x_1^{k-1} + x_1^k.$$

776. $x_1 + x_2 = f_1 - x_3$,

$$(f_1 - x_1)(f_1 - x_2)(f_1 - x_3) = f_1^3 - f_1^3 + f_1 f_2 - f_3 = f_1 f_2 - f_3,$$
$$2x_1 - x_2 - x_3 = 3x_1 - f_1,$$
$$(3x_1 - f_1)(3x_2 - f_1)(3x_3 - f_1) = 27 f_3 - 9 f_1 f_2 + 2 f_1^3,$$
$$x_1^2 + x_1 x_2 + x_2^2 = f_1^2 - f_2 - f_1 x_3,$$
$$x_1^2 - x_2 x_3 = f_1 x_1 - f_2.$$

777. $\sum\limits_{i=1}^{n} \dfrac{\partial f_k}{\partial x_i} = (n-k) f_{k-1}$.

778. Let $F(x_1, x_2, \ldots, x_n) = \Phi(f_1, f_2, \ldots, f_n)$. Then

$$\sum_{i=1}^{n} \frac{\partial F}{\partial x_i} = n \frac{\partial \Phi}{\partial f_1} + (n-1) f_1 \frac{\partial \Phi}{\partial f_2} + \ldots + f_{n-1} \frac{\partial \Phi}{\partial f_n}.$$

779. Let $\varphi(a) = F(x_1 + a, x_2 + a, \ldots, x_n + a)$. Then

$$\varphi'(a) = \sum_{i=1}^{n} \frac{\partial F(x_1 + a, x_2 + a, \ldots, x_n + a)}{\partial x_i}.$$

Since $\varphi(a)$ is not dependent on a, then $\varphi'(a)$ is identically zero, whence it follows that $\sum\limits_{i=1}^{n} \dfrac{\partial F}{\partial x_i} = 0$. Conversely, if $\sum\limits_{i=1}^{n} \dfrac{\partial F(x_1, x_2, \ldots, x_n)}{\partial x_i}$ is identically zero, then $\varphi'(a) = \sum\limits_{i=1}^{n} \dfrac{\partial F(x_1 + a, \ldots, x_n + a)}{\partial x_i} = 0$ whence it follows that $\varphi(a)$ does not depend on a and $\varphi(a) = \varphi(0)$, that is,

$$F(x_1 + a, x_2 + a, \ldots, x_n + a) = F(x_1, x_2, \ldots, x_n).$$

By virtue of the preceding problem, the condition $\sum\limits_{i=1}^{n} \dfrac{\partial F}{\partial x_i} = 0$ is equivalent to the condition

$$n \frac{\partial \Phi}{\partial f_1} + (n-1) f_1 \frac{\partial \Phi}{\partial f_2} + \ldots + f_{n-1} \frac{\partial \Phi}{\partial f_n} = 0.$$

780. Let $F(x_1, x_2, \ldots, x_n)$ be a homogeneous symmetric polynomial of degree two. Then its expression in terms of the elementary symmetric polynomials is of the form $\Phi = Af_1^2 + Bf_2$. By virtue of the result of Problem 779, it must be true that $n \cdot 2Af_1 + (n-1)Bf_1 = 0$, whence $A = (n-1)\alpha$, $B = -2n\alpha$ and

$$F(x_1, x_2, \ldots, x_n) = \alpha[(n-1)f_1^2 - 2nf_2] = \alpha \sum_{i<k} (x_i - x_k)^2.$$

781. The expression of a homogeneous symmetric polynomial of degree three in terms of the elementary polynomials is of the form $Af_1^3 + Bf_1f_2 + Cf_3$. By virtue of the result of Problem 779, it must be true that $3Anf_1^2 + nBf_2 + (n-1)Bf_1^2 + (n-2)Cf_2 = 0$ whence

$$F(x_1, x_2, \ldots, x_n) = \alpha[(n-1)(n-2)f_1^3 - 3n(n-2)f_1f_2 + 3n^2f_3].$$

782. $(n-2)f_1^2f_2^2 - 2(n-1)f_1^3f_3 - 4(n-2)f_2^3$

$$+(10n-12)f_1f_2f_3 - 4(n-1)f_1^2f_4 - 9nf_3^2 + 8nf_2f_4.$$

783. We can take

$$\varphi_k = f_k\left(x_1 - \frac{f_1}{n},\ x_2 - \frac{f_1}{n},\ \ldots,\ x_n - \frac{f_1}{n}\right).$$

Each function φ_k has the required property. Furthermore, if $F(x_1, x_2, \ldots, x_n) = F(x_1+a, x_2+a, \ldots, x_n+a)$ and $F(x_1, x_2, \ldots, x_n) = \Phi(f_1, f_2, \ldots, f_n)$, then

$$F(x_1, x_2, \ldots, x_n) = \Phi(0, \varphi_2, \varphi_3, \ldots, \varphi_n).$$

784. (a) $-4\varphi_2^3 - 27\varphi_3^2$, (b) $18\varphi_2^2$.

785. (a) $8\varphi_3$,

(b) $-4\varphi_2^3\varphi_3^2 + 16\varphi_2^4\varphi_4 - 27\varphi_3^4 + 144\varphi_2\varphi_3^2\varphi_4 - 128\varphi_2^2\varphi_4^2 + 256\varphi_4^3$.

786. $s_2 = f_1^2 - 2f_2$;

$s_3 = f_1^3 - 3f_1f_2 + 3f_3$;

$s_4 = f_1^4 - 4f_1^2f_2 + 2f_2^2 + 4f_1f_3 - 4f_4$;

$s_5 = f_1^5 - 5f_1^3f_2 + 5f_1f_2^2 + 5f_1^2f_3 - 5f_2f_3 - 5f_1f_4 + 5f_5$;

$s_6 = f_1^6 - 6f_1^4f_2 + 9f_1^2f_2^2 + 6f_1^3f_3 - 2f_2^3 - 12f_1f_2f_3$

$$-6f_1^2f_4 + 3f_3^2 + 6f_2f_4 + 6f_1f_5 - 6f_6.$$

787. $2f_2 = s_1^2 - s_2$;

$6f_3 = s_1^3 - 3s_1s_2 + 2s_3$;

$24f_4 = s_1^4 - 6s_1^2s_2 + 8s_1s_3 + 3s_2^2 - 6s_4$;

$120f_5 = s_1^5 - 10s_1^3s_2 + 20s_1^2s_3 + 15s_1s_2^2 - 20s_2s_3 - 30s_1s_4 + 24s_5$;

$720f_6 = s_1^6 - 15s_1^4s_2 + 40s_1^3s_3 + 45s_1^2s_2^2 - 120s_1s_3s_3$

$$-15s_2^3 - 90s_1^2s_4 + 40s_3^2 + 90s_2s_4 + 144s_1s_5 - 120s_6.$$

788. $s_5 = 859$. **789.** $s_8 = 13$. **790.** $s_{10} = 621$.

791. $s_1 = -1$, $s_2 = s_3 = \ldots = s_n = 0$.

792. This is readily proved by mathematical induction by means of the relation

$$as_k + bs_{k-1} + cs_{k-2} = 0$$

where $s_k = x_1^k + x_2^k$.

793. $s_5-s_1^5=5(f_1^2-f_2)(f_3-f_1f_2);\quad s_3-s_1^3=3(f_3-f_1f_2)$.

794. $s_5=-5f_2f_3;\ s_3=3f_3;\ s_2=-2f_2$.

795. $s_7=-7f_2f_5;\ s_2=-2f_2;\ s_5=5f_5$.

796. $x^n-a=0$.

797. $x^n-\frac{a}{1}x^{n-1}+\frac{a^2}{1\cdot 2}x^{n-2}-\ldots+(-1)^n\frac{a^n}{n!}=0$.

798. $x^n+\frac{P_1(\alpha)}{1}x^{n-1}+\frac{P_2(\alpha)}{2!}x^{n-2}+\ldots+\frac{P_n(\alpha)}{n!}=0$, where $P_1, P_2, \ldots, P_n$ are Hermite polynomials: $P_k(x)=(-1)^k e^{\frac{x^2}{2}}\frac{d^k e^{-\frac{x^2}{2}}}{dx^k}$, α is a root of the Hermite polynomial $P_{n+1}(x)$.

Solution. Let the desired equation have the form

$$x^n+a_1x^{n-1}+a_2x^{n-2}+\ldots+a_n=0.$$

By virtue of Newton's formulas

$$\begin{aligned} a_1 &=\alpha,\\ 2a_2 &=\alpha a_1-1,\\ 3a_3 &=\alpha a_2-a_1,\\ &\cdots\cdots\cdots\cdots\\ ka_k &=\alpha a_{k-1}-a_{k-2},\\ &\cdots\cdots\cdots\cdots\\ na_n &=\alpha a_{n-1}-a_{n-2},\\ 0 &=\alpha a_n-a_{n-1}. \end{aligned}$$

From these relations it follows that a_k is a polynomial of degree k in α. Set $k!a_k=P_k(\alpha)$. Then, taking $P_0=1$, we get

$$P_1=\alpha \text{ and } P_k-\alpha P_{k-1}+(k-1)P_{k-2}=0,$$
$$-\alpha P_n+nP_{n-1}=0.$$

The first relations show that P_k is a Hermite polynomial in α (see Problem 707). The latter yields $P_{n+1}(\alpha)=0$.

799. $\frac{1}{2}(s_k^2-s_{2k})$.

800. $\sum_{i=1}^{n}(x+x_i)^k=\sum_{m=0}^{k}C_k^m s_{k-m}x^m$,

$$\sum_{i=1}^{n}\sum_{j=1}^{n}(x_j+x_i)^k=\sum_{m=0}^{k}C_k^m s_{k-m}s_m,$$

$$\sum_{i<j}(x_i+x_j)^k=\frac{1}{2}\left(\sum_{m=0}^{k}C_k^m s_{k-m}s_m-2^k s_k\right).$$

801. $\sum_{i<j} (x_i-x_j)^{2k} = \frac{1}{2} \sum_{m=0}^{2k} C_{2k}^{m} (-1)^m s_m s_{2k-m}.$

802. Multiply the second column by $-s_1$, the third by s_2, ..., the kth by $(-1)^{k-1} s_{k-1}$ and add to the first. By Newton's formulas, we get

$$\begin{vmatrix} f_1 & 1 & 0 & & 0 \\ 2f_2 & f_1 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ (k-1)f_{k-1} & f_{k-2} & & \dots f_1 & 1 \\ kf_k & f_{k-1} & & & f_1 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & f_1 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & f_{k-2} & f_{k-3} & \dots & 1 \\ (-1)^{k-1} s_k & f_{k-1} & f_{k-2} & \dots & f_1 \end{vmatrix} = s_k.$$

803. Multiply the second column by $-f_1$, the third by f_2, ..., the kth by $(-1)^{k-1}f_{k-1}$ and add the results to the first column. Newton's formulas yield the desired result.

804. $n!\,(x^n - f_1 x^{n-1} + f_2 x^{n-2} + \dots + (-1)^n f_n)$.

805. $\dfrac{\varphi(n)}{\varphi\left(\dfrac{n}{d}\right)} \mu\left(\dfrac{n}{d}\right)$ where d is the greatest common divisor of m and n.

806. By virtue of the result of Problems 117, 119, it suffices to consider the case $n=p_1 p_2 \dots p_k$, where $p_1, p_2, \dots, p_k$ are distinct odd primes. In this case, $s_1=s_2=s_4=(-1)^k$; $s_3=2(-1)^{k-1}$ if n is divisible by 3, and $s_3=(-1)^k$ if n is not divisible by 3. Computations by Newton's formulas yield:

$$f_2 = \frac{1-(-1)^k}{2},$$

$$f_3 = \frac{(-1)^{k-1}-1}{2} \quad \text{if } n \text{ is divisible by 3,}$$

$$f_3 = \frac{(-1)^k-1}{2} \quad \text{if } n \text{ is not divisible by 3,}$$

$$f_4 = \frac{(-1)^{k-1}-1}{2} \quad \text{if } n \text{ is divisible by 3,}$$

$$f_4 = \frac{(-1)^{k-1}+1}{2} \quad \text{if } n \text{ is not divisible by 3.}$$

807. $s_1=s_2=s_3= \dots =s_n=a$. Hence, for $k \leqslant n$

$$kf_k = af_{k-1} - af_{k-2} + \dots + (-1)^{k-1} f_1,$$
$$(k-1)\,f_{k-1} = af_{k-2} + \dots + (-1)^{k-2} f_1,$$

whence

$$kf_k = (a-k+1)\,f_{k-1}, \quad f_k = \frac{a-k+1}{k} f_{k-1}.$$

Obviously, $f_1=a$; therefore,

$$f_2=\frac{a(a-1)}{1\cdot 2},\quad \ldots,\quad f_k=\frac{a(a-1)\ \ldots\ (a-k+1)}{1\cdot 2\ \ldots\ k}$$

and so $x_1, x_2, \ldots, x_n$ are roots of the equation

$$x^n-\frac{a}{1}x^{n-1}+\frac{a(a-1)}{1\cdot 2}x^{n-2}-\ldots+(-1)^n\frac{a(a-1)\ \ldots\ (a-n+1)}{n!}=0,$$

$$s_{n+1}=a-\frac{a(1-a)(2-a)\ \ldots\ (n-a)}{n!}.$$

808. $(x-a)(x-b)[x^n+(a+b)x^{n-1}+\ldots+(a^n+a^{n-1}b+\ldots+b^n)]=$
$=(x-a)[x^{n+1}+ax^n+a^2x^{n-1}+\ldots+a^nx-b(a^n+a^{n-1}b+\ldots+b^n)]$
$=x^{n+2}-(a^{n+1}+a^nb+\ldots+b^{n+1})x+ab(a^n+a^{n-1}b+\ldots+b^n).$

The power sums $\sigma_1, \sigma_2, \ldots, \sigma_n$ for the new polynomial are obviously equal to zero. But $\sigma_k=s_k+a^k+b^k$. Hence, $s_k=-(a^k+b^k)$ for $1\leqslant k\leqslant n$.

809. $s_k=-a^k-b^k$ for odd k,

$$s_k=-\left(a^{\frac{k}{2}}-b^{\frac{k}{2}}\right)^2 \quad \text{for even } k.$$

810. (a) $(x+a)(x^2+ax+b)-c=0$;

(b) $x(x-a^2+3b)^2-(a^2b^2-4a^3c-4b^3+18abc-27c^2)=0$;

(c) $x^3+(3b-a^2)x^2+b(3b-a^2)x+b^3-a^3c=0$;

(d) $x^2(x-a^2+3b)+(a^2b^2-4a^3c-4b^3+18abc-27c^2)=0$;

(e) $x^3-(a^2-2b)x^2+(b^2-2ac)x-c^2=0$;

(f) $x^3+(a^3-3ab+3c)x^2+(b^3-3abc+3c^2)x+c^3=0$.

811. $y^2+(2a^3-9ab+27c)y+(a^2-3b)^3=0$.

812. $y^2-\dfrac{ab-3c}{c}y+\dfrac{b^3+a^3c-6abc+9c^2}{c^2}=0$.

813. $y^6-\dfrac{ab-3c}{c}y^5+\dfrac{b^3-5abc+6c^2}{c^2}y^4$

$$-\frac{a^2b^2-2b^3-2a^3b+6abc-7c^2}{c^2}y^3+\frac{b^3-5abc+6c^2}{c^2}y^2-\frac{ab-3c}{c}y+1=0.$$

814. (a) $y^3-by^2+(ac-4d)y-(a^2d+c^2-4bd)=0$

(Ferrari's resolvent),

(b) $y^3-(3a^2-8b)y^2+(3a^4-16a^2b+16b^2+16ac-64d)y-(a^3$
$-4ab+8c)^2=0$

(Euler's resolvent),

(c) $y^6-by^5+(ac-d)y^4-(a^2d+c^2-2bd)y^3+d(ac-d)y^2-bd^2y$
$+d^3=0$;

(d) $y^6+3ay^5+(3a^2+2b)y^4+(a^3+4ab)y^3+(2a^2b+b^2+ac-4d)y^2$
$+(ab^2+a^2c-4ad)y+(abc-a^2d-c^2)=0.$

815. $x=\dfrac{-a\pm\sqrt{a^2-4b+4y_1}\pm\sqrt{a^2-4b+4y_2}\pm\sqrt{a^2-4b+4y_3}}{4}$.

The signs of the square roots must be taken so that their product is equal to $-a^3+4ab-8c$.

816.

$$x=\frac{\pm\sqrt{4a+\sqrt[3]{b^2-64a^3}}\pm\sqrt{4a+\varepsilon\sqrt[3]{b^2-64a^3}}\pm\sqrt{4a+\varepsilon^2\sqrt[3]{b^2-64a^3}}}{2},$$

$$\varepsilon=-\frac{1}{2}+\frac{i\sqrt{3}}{2}.$$

The signs of the square roots are taken so that their product is equal to $-b$.

817. $(y+a)^4(y^2+6ay+25a^2)+3125b^4y=0$.

Solution. The roots of the desired equation are:

$$y_1=(x_1x_2+x_2x_3+x_3x_4+x_4x_5+x_5x_1)\ (x_1x_3+x_3x_5+x_5x_2+x_2x_4+x_4x_1);$$
$$y_2=(x_1x_3+x_3x_2+x_2x_5+x_5x_4+x_4x_1)\ (x_1x_2+x_2x_4+x_4x_3+x_3x_5+x_5x_1);$$
$$y_3=(x_5x_2+x_2x_4+x_4x_3+x_3x_1+x_1x_5)\ (x_5x_4+x_4x_1+x_1x_2+x_2x_3+x_3x_5);$$
$$y_4=(x_2x_1+x_1x_3+x_3x_5+x_5x_4+x_4x_2)\ (x_2x_3+x_3x_4+x_4x_1+x_1x_5+x_5x_2);$$
$$y_5=(x_5x_3+x_3x_2+x_2x_4+x_4x_1+x_1x_5)\ (x_5x_2+x_2x_1+x_1x_3+x_3x_4+x_4x_5);$$
$$y_6=(x_2x_1+x_1x_4+x_4x_3+x_3x_5+x_5x_2)\ (x_2x_4+x_4x_5+x_5x_1+x_1x_3+x_3x_2).$$

The sought-for equation is obviously of the form

$$y^6+c_1ay^5+c_2a^2y^4+c_3a^3y^3+c_4a^4y^2+(c_5a^5+c_6b^4)\,y+(c_7a^6+c_8ab^4)=0,$$

where $c_1, c_2, \ldots, c_8$ are absolute constants. To determine them, put $a=-1$ $b=0$, and $a=0$, $b=-1$. We get

a	b	x_1	x_2	x_3	x_4	x_5	y_1	y_2	y_3	y_4	y_5	y_6
-1	0	1	i	-1	$-i$	0	1	$3-4i$	1	1	$3+4i$	1
0	-1	1	ε	ε^2	ε^3	ε^4	0	-5	$-5\varepsilon^4$	$-5\varepsilon^3$	$-5\varepsilon^2$	-5ε

In the first case, the desired equation is of the form

$$(y-1)^4(y^2-6y+25)=0.$$

In the second case, $y^6+3125y=0$, whence we determine all the coefficients, except c_8. It is easy to verify that $c_8=0$. To do this, we can, say, take $a=-5$, $b=4$. In this case, $x_1=x_2=1$, and the remaining roots satisfy the equation $x^3+2x^2+3x+4=0$ and all the necessary computations are performed with ease.

818. Let $f(x)=(x-x_1)(x-x_2)\dots(x-x_n)$, where $x_1, x_2, \dots, x_n$ are independent variables. Also, let

$$x^{k-1}\varphi(x)=f(x)\,q_k(x)+r_k(x) \quad \text{and} \quad r_k(x)=c_{k1}+c_{k2}x+\dots+c_{kn}x^{n-1}.$$

The coefficients c_{ks} are obviously some polynomials in $x_1, x_2, \dots, x_n$. Furthermore,

$$\begin{vmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \dots & \dots & \dots & \dots \\ c_{n1} & c_{n2} & \dots & c_{nn} \end{vmatrix} \cdot \begin{vmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_n \\ \dots & \dots & \dots & \dots \\ x_1^{n-1} & x_2^{n-1} & \dots & x_n^{n-1} \end{vmatrix}$$

$$= \begin{vmatrix} r_1(x_1) & r_1(x_2) & \dots & r_1(x_n) \\ r_2(x_1) & r_2(x_2) & \dots & r_2(x_n) \\ \dots & \dots & \dots & \dots \\ r_n(x_1) & r_n(x_2) & \dots & r_n(x_n) \end{vmatrix}$$

$$= \begin{vmatrix} \varphi(x_1) & \varphi(x_2) & \dots & \varphi(x_n) \\ x_1\varphi(x_1) & x_2\varphi(x_2) & \dots & x_n\varphi(x_n) \\ \dots & \dots & \dots & \dots \\ x_1^{n-1}\varphi(x_1) & x_2^{n-1}\varphi(x_2) & \dots & x_n^{n-1}\varphi(x_n) \end{vmatrix}$$

$$=\varphi(x_1)\,\varphi(x_2)\dots\varphi(x_n) \begin{vmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_n \\ \dots & \dots & \dots & \dots \\ x_1^{n-1} & x_2^{n-1} & \dots & x_n^{n-1} \end{vmatrix}$$

whence it follows that

$$\begin{vmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \dots & \dots & \dots & \dots \\ c_{n1} & c_{n2} & \dots & c_{nn} \end{vmatrix} = \varphi(x_1)\,\varphi(x_2)\dots\varphi(x_n)=R(f, \varphi).$$

The last equation is an identity between the polynomials in the independent variables $x_1, x_2, \dots, x_n$ and therefore remains true for all particular values of these variables.

819. First of all, satisfy yourself that all polynomials $\psi_k(x)$ are of degree $n-1$. Introduce the following notations:

$$f_k(x)=a_0x^{k-1}+\dots+a_{k-1},$$

$$\bar{f}_k(x)=a_kx^{n-k}+\dots+a_n,$$

$$\varphi_k(x)=b_0x^{k-1}+\dots+b_{k-1},$$

$$\bar{\varphi}_k(x)=b_kx^{n-k}+\dots+b_n.$$

Then

$$f(x)=x^{n-k+1}f_k(x)+\bar{f}_k(x),$$

$$\varphi(x)=x^{n-k+1}\varphi_k(x)+\bar{\varphi}_k(x),$$

$$\psi_k(x)=f_k(x)\,[x^{n-k+1}\varphi_k(x)+\bar{\varphi}_k(x)]-\varphi_k(x)\,[x^{n-k+1}f_k(x)$$
$$+\bar{f}_k(x)]=f_k(x)\,\bar{\varphi}_k-\varphi_k\bar{f}_k(x)=(a_0b_k-b_0a_k)\,x^{n-1}+\dots$$

Suppose $\psi_k(x)=c_{k1}+c_{k2}x+\dots+c_{kn}x^{n-1}$ and let $x_1, x_2, \dots, x_n$ be the roots of the polynomial $f(x)$. Then

$$\begin{vmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \dots & \dots & \dots & \dots \\ c_{n1} & c_{n2} & \dots & c_{nn} \end{vmatrix} \cdot \begin{vmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_n \\ \dots & \dots & \dots & \dots \\ x_1^{n-1} & x_2^{n-1} & \dots & x_n^{n-1} \end{vmatrix}$$

$$= \begin{vmatrix} \psi_1(x_1) & \psi_1(x_2) & \dots & \psi_1(x_n) \\ \psi_2(x_1) & \psi_2(x_2) & \dots & \psi_2(x_n) \\ \dots & \dots & \dots & \dots \\ \psi_n(x_1) & \psi_n(x_2) & \dots & \psi_n(x_n) \end{vmatrix}$$

$$= \begin{vmatrix} f_1(x_1)\varphi(x_1) & f_1(x_2)\varphi(x_2) & \dots & f_1(x_n)\varphi(x_n) \\ f_2(x_1)\varphi(x_1) & f_2(x_2)\varphi(x_2) & \dots & f_2(x_n)\varphi(x_n) \\ \dots & \dots & \dots & \dots \\ f_n(x_1)\varphi(x_1) & f_n(x_2)\varphi(x_2) & \dots & f_n(x_n)\varphi(x_n) \end{vmatrix}$$

$$= \varphi(x_1)\varphi(x_2)\dots\varphi(x_n)\cdot \begin{vmatrix} f_1(x_1) & f_1(x_2) & \dots & f_1(x_n) \\ f_2(x_1) & f_2(x_2) & \dots & f_2(x_n) \\ \dots & \dots & \dots & \dots \\ f_n(x_1) & f_n(x_2) & \dots & f_n(x_n) \end{vmatrix}$$

$$= \varphi(x_1)\varphi(x_2)\dots\varphi(x_n) \begin{vmatrix} a_0 & 0 & \dots & 0 \\ a_1 & a_0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ a_{n-1} & a_{n-2} & \dots & a_0 \end{vmatrix} \cdot \begin{vmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_n \\ \dots & \dots & \dots & \dots \\ x_1^{n-1} & x_2^{n-1} & \dots & x_n^{n-1} \end{vmatrix}$$

$$= \varphi(x_1)\varphi(x_2)\dots\varphi(x_n)\,a_0^n \cdot \begin{vmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_n \\ \dots & \dots & \dots & \dots \\ x_1^{n-1} & x_2^{n-1} & \dots & x_n^{n-1} \end{vmatrix}$$

whence it follows that

$$\begin{vmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \dots & \dots & \dots & \dots \\ c_{n1} & c_{n2} & \dots & c_{nn} \end{vmatrix} = a_0^n \varphi(x_1)\varphi(x_2)\dots\varphi(x_n) = R(f, \varphi).$$

820. The polynomials χ_k have degree not above $n-1$. This is obvious for $1 \leqslant k \leqslant n-m$, and for $k > n-m$ it follows from the fact that χ_k are Bezout polynomials ψ_{k-n+m} for $f(x)$ and $x^{n-m}\varphi(x)$. Let $\chi_k(x)=c_{k1}+c_{k2}x+\dots+c_{kn}x^{n-1}$ and

$$\Delta = \begin{vmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_n \\ \dots & \dots & \dots & \dots \\ x_1^{n-1} & x_2^{n-1} & \dots & x_n^{n-1} \end{vmatrix}.$$

Then

$$\begin{vmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \dots & \dots & \dots & \dots \\ c_{n1} & c_{n2} & \dots & c_{nn} \end{vmatrix} \cdot \Delta = \begin{vmatrix} \chi_1(x_1) & \chi_1(x_2) & \dots & \chi_1(x_n) \\ \chi_2(x_1) & \chi_2(x_2) & \dots & \chi_2(x_n) \\ \dots & \dots & \dots & \dots \\ \chi_n(x_1) & \chi_n(x_2) & \dots & \chi_n(x_n) \end{vmatrix}$$

$$= \varphi(x_1)\varphi(x_2)\dots\varphi(x_n)$$

$$\times\begin{vmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_n \\ \dots & \dots & \dots & \dots \\ x_1^{n-m-1} & x_2^{n-m-1} & \dots & x_n^{n-m-1} \\ x_1^{n-m} f_0(x_1) & x_2^{n-m} f_0(x_2) & \dots & x_n^{n-m} f_0(x_n) \\ \dots & \dots & \dots & \dots \\ x_1^{n-m} f_{m-1}(x_1) & x_2^{n-m} f_{m-1}(x_2) & \dots & x_n^{n-m} f_{m-1}(x_n) \end{vmatrix}$$

$$=\varphi(x_1)\varphi(x_2)\dots\varphi(x_n)\Delta\cdot\begin{vmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & \ddots & & & & \\ & & & 1 & & & \\ & & & & a_0 & & \\ & & & & a_1 & a_0 & \\ & & & & \dots & \dots & \\ & & & & a_{m-1} & a_{m-2} & \dots & a_0 \end{vmatrix}$$

$$=a_0^m\,\varphi(x_1)\dots\varphi(x_n)\,\Delta$$

whence immediately follows the desired result.

821. (a) -7, (b) 243, (c) 0, (d) -59, (e) 4854, (f) $(b_0a_2-b_2a_0)^2-(b_0a_1-$ $-b_1a_0)(b_1a_2-b_2a_1)$.

822. (a) For $\lambda=3$ and $\lambda=-1$;

(b) $\lambda=1$, $\lambda=\dfrac{-2+\sqrt{2}\pm\sqrt{4\sqrt{2}-2}}{2}$,

$\lambda=\dfrac{-2-\sqrt{2}\pm i\sqrt{4\sqrt{2}+2}}{2}$;

(c) $\lambda=\pm\sqrt{-2}$, $\lambda=\pm\sqrt{-12}$,

823. (a) $y^6-4y^4+3y^2-12y+12=0$,

(b) $5y^5-7y^4+6y^3-2y^2-y-1=0$,

(c) $y^3+4y^2-y-4=0$.

824. (a) $x_1=1$, $x_2=2$, $x_3=0$, $x_4=-2$,
$y_1=2$, $y_2=3$, $y_3=-1$, $y_4=1$.

(b) $x_1=0$, $x_2=3$, $x_3=2$, $x_4=2$,
$y_1=1$, $y_2=0$, $y_3=2$, $y_4=-1$.

(c) $x_1=x_2=1$, $x_3=-1$, $x_4=2$,
$y_1=y_2=-1$; $y_3=1$; $y_4=2$.

(d) $x_1=0$, $x_2=0$, $x_3=1$, $x_4=1$, $x_{5,6}=2$,
$y_1=1$, $y_2=3$, $y_3=2$, $y_4=3$, $y_{5,6}=1\pm i\sqrt{2}$.

(e) $x_1=0$, $x_2=0$, $x_3=2$, $x_4=x_5=2$, $x_6=-4$,
$y_1=2$, $y_2=-2$, $y_3=0$, $y_4=y_5=2$, $y_6=2$,
$x_7=4$, $x_8=-6$, $x_9=-2/3$, $y_7=6$, $y_8=4$, $y_9=4/3$.

825. $a_0^n a_n^{n-1}$.

826. Let $f(x)=a_0(x-x_1)(x-x_2)\dots(x-x_n)$;

$$\varphi_1(x)=b_0x^k+\dots+b_k;\ \varphi_2(x)=c_0x^m+\dots+c_m.$$

Then

$$R(f,\ \varphi_1\cdot\varphi_2)=a_0^{m+k}\prod_{i=1}^{n}\varphi_1(x_i)\,\varphi_2(x_i)$$

$$=\left[a_0^m\prod_{i=1}^{n}\varphi_1(x_i)\right]\left[\,{}_0^k\prod_{i=1}^{n}\varphi_2(x_i)\right]=R(f,\ \varphi_1)\cdot R(f,\ \varphi_2).$$

827. Only the case $n>2$ is of interest. Denote by d the greatest common divisor of m and n; $\xi_1,\ \xi_2,\ \dots$ are primitive nth roots of unity; $\eta_1,\ \eta_2,\ \dots$ are primitive roots of unity of degree $\frac{n}{d}=n_1$. Then,

$$R(X_n,\ x^m-1)=\prod(\xi_i^m-1)=\prod(1-\xi_i^m)$$

$$=\left[\prod(1-\eta_i)\right]^{\frac{\varphi(n)}{\varphi(n_1)}}[X_{n_1}(1)]^{\frac{\varphi(n)}{\varphi(n_1)}}$$

If m is divisible by n, then $R(X_n,\ x^m-1)=0$. But if m is not divisible by n, then $n_1\neq 1$, and, by virtue of Problem 123, $X_{n_1}(1)=1$ for $n_1\neq p^\lambda$, $X_{n_1}(1)=p$ for $n_1=p^\lambda$ (p is prime). And so

$$R(X_n,\ x^m-1)=0\ \text{ for }\ n_1=\frac{n}{d}=1,$$

$$R(X_n,\ x^m-1)=p^{\frac{\varphi(n)}{\varphi(n_1)}}\ \text{ for }\ n_1=\frac{n}{d}=p^\lambda,$$

$$R(X_n,\ x^m-1)=1\ \text{ in all other cases.}$$

828. It is obvious that $R(X_n,\ X_m)$ is a positive integer which is a divisor of $R(X_n,\ x^m-1)$ and of $R(X_m,\ x^n-1)$. Denote by d the greatest common divisor of m and n. If m is not divisible by n, and n is not divisible by m, then $\frac{m}{d}$ and $\frac{n}{d}$ are different from 1 and are relatively prime. By virtue of the result of the preceding problem, $R(X_n,\ x^m-1)$ and $R(X_m,\ x^n-1)$ are in this case relatively prime, and therefore $R(X_n,\ X_m)=1$.

It remains to consider the case when one of the numbers m, n is divisible by the other. For definiteness, say n divides m.

If $m=n$, then $R(X_m,\ X_n)=0$. If $\frac{m}{n}$ is not a power of a prime, then $R(X_m, x^n-1)=1$ and, hence, $R(X_m,\ X_n)=1$. Finally, suppose that $m=np^\lambda$. Then

$$R(X_m,\ X_n)=\prod_{\delta/n}R(X_m,\ x^\delta-1)^{\mu\left(\frac{n}{\delta}\right)}.$$

All factors on the right are equal to unity, except those for which $\frac{m}{\delta}$ is a power of the number p.

If n is not divisible by p, then there is only one factor different from unity when $\delta=n$ and

$$R(X_m, X_n)=R(X_m, x^n-1)=p^{\frac{\varphi(m)}{\varphi(m/n)}}=p^{\varphi(n)}.$$

If n is divisible by p, then there are two factors different from unity: when $\delta=n$ and $\delta=\frac{n}{p}$. Then

$$R(X_m, X_n)=\frac{R(X_m, x^n-1)}{R(X_m, x^{n/p}-1)}=p^{\frac{\varphi(m)}{\varphi(m/n)}-\frac{\varphi(m)}{\varphi(mp/n)}}$$

$$=p^{\varphi(m)\left[\frac{1}{p^{\lambda-1}(p-1)}-\frac{1}{p^{\lambda}(p-1)}\right]}=p^{\frac{\varphi(m)}{p^{\lambda}}}=p^{\varphi(n)}.$$

Thus,

$$R(X_m, X_n)=0 \text{ when } m=n,$$

$$R(X_m, X_n)=p^{\varphi(n)} \text{ when } m=np^{\lambda},$$

$$R(X_m, X_n)=1 \text{ otherwise.}$$

829. (a) 49, (b) -107, (c) -843, (d) 725, (e) 2777.

830. (a) $3125\,(b^2-4a^5)^2$, (b) $\lambda^4(4\lambda-27)^3$,

(c) $(b^2-3ab+9a^2)^2$, (d) $4\,(\lambda^2-8\lambda+32)^3$.

831. (a) $\lambda=\pm 2$; (b) $\lambda_1=3$, $\lambda_{2,3}=3\left(-\frac{1}{2}\pm\frac{i\sqrt{3}}{2}\right)$;

(c) $\lambda_1=0$, $\lambda_2=-3$, $\lambda_3=125$;

(d) $\lambda_1=-1$, $\lambda_2=-\frac{3}{2}$, $\lambda_{3,4}=\frac{7}{2}\pm\frac{9}{2}i\sqrt{3}$.

832. In the general case, if the discriminant is positive, then the number of pairs of conjugate complex roots is even, if the discriminant is negative, then it is odd.

In particular, for a third-degree polynomial, if $D>0$, then all roots are real; if $D<0$, then two roots are complex conjugates.

For a fourth-degree polynomial for $D>0$, either all roots are real or all roots are complex. For $D<0$ there are two real roots and one pair of conjugate complex roots.

833. $f=x^n+a$, $f'=nx^{n-1}$,

$$R(f', f)=n^n a^{n-1},\quad D(f)=(-1)^{\frac{n(n-1)}{2}}n^n a^{n-1}.$$

834. $f=x^n+px+q$, $f'=nx^{n-1}+p$,

$$R(f',f)=n^n\prod_{k=0}^{n-2}\left(q+\frac{n-1}{n}p\sqrt[n-1]{-\frac{p}{n}}\,\varepsilon^k\right)$$

where $\varepsilon=\cos\frac{2\pi}{n-1}+i\sin\frac{2\pi}{n-1}$.

$$R(f',f)=n^n\left[q^{n-1}+\frac{(n-1)^{n-1}p^{n-1}}{n^{n-1}}\cdot\left(-\frac{p}{n}\right)(-1)^{n-2}\right]$$

$$=n^n q^{n-1}+(-1)^{n-1}(n-1)^{n-1}p^n,$$

$$D(f)=(-1)^{\frac{n(n-1)}{2}}n^n q^{n-1}+(-1)^{\frac{(n-1)(n-2)}{2}}(n-1)^{n-1}p^n.$$

835. Let the greatest common divisor of m and n be d. Introduce the notations: $m_1=\frac{m}{d}$, $n_1=\frac{n}{d}$, ε is a primitive nth root of unity, η is a primitive n_1th root of 1, $a_0x^{m+n}+a_1x^m+a_2=f(x)$. Then $f'(x)=(m+n)a_0x^{m+n-1}+ma_1x^{m-1}$. The roots of the derivative are $\xi_1=\xi_2=\ldots=\xi_{m-1}=0$,

$$\xi_{m+k}=\sqrt[n]{-\frac{ma_1}{(m+n)a_0}}\,\varepsilon^k=\xi_m\varepsilon^k,\ k=0,\ 1,\ \ldots,\ n-1.$$

Furthermore,

$$R(f',f)=(m+n)^{m+n}a_0^{m+n}a_2^{m-1}\prod_{k=0}^{n-1}\left[a_2+\frac{na_1}{m+n}\xi_m^m\varepsilon^{km}\right]$$

$$=(m+n)^{m+n}a_0^{m+n}a_2^{m-1}\left[\prod_{k=0}^{n_1-1}\left(a_2+\frac{na_1}{m+n}\right)\xi_m\eta^k\right]^d$$

$$=(m+n)^{m+n}a_0^{m+n}a_2^{m-1}\left[a_2^{n_1}+(-1)^{m_1+n_1-1}\frac{n^{n_1}m^{m_1}a_1^{m_1+n_1}}{(m+n)^{m_1+n_1}a_0^{m_1}}\right]^d$$

$$=a_0^n a_2^{m-1}[(m+n)^{m_1+n_1}a_0^{m_1}a_2^{n_1}+(-1)^{m_1+n_1-1}n^{n_1}m^{m_1}a_1^{m_1+n_1}]^d$$

and, hence,

$$D(f)=(-1)^{\frac{(m+n)(m+n-1)}{2}}a_0^{n-1}a_2^{m-1}[(m+n)^{m_1+n_1}a_0^{m_1}a_2^{n_1}$$
$$+(-1)^{m_1+n_1-1}n^{n_1}m^{m_1}a_1^{m_1+n_1}]^d.$$

836. The discriminants are equal.

837. $x_1x_2+x_3x_4-x_1x_3-x_2x_4=(x_1-x_4)(x_2-x_3)$,

$x_1x_3+x_2x_4-x_1x_4-x_2x_3=(x_1-x_2)(x_3-x_4)$,

$x_1x_4+x_2x_3-x_1x_2-x_3x_4=(x_1-x_3)(x_4-x_2)$.

Squaring and multiplying these equations, we get the required result.

838. Let $f(x)=a_0(x-x_1)(x-x_2)\dots(x-x_n)$. Then

$$D(f(x)\ (x-a))=a_0^{2n}(a-x_1)^2(a-x_2)^2\dots(a-x_n)^2\prod_{i<k}(x_i-x_k)^2=D\left(f(x)\right)[f(a)]^2.$$

839. Let us denote $\varphi(x)=x^{n-1}+x^{n-2}+\dots+1$. Then $(x-1)\varphi(x)=x^n-1$, whence it follows that

$$D(\varphi)[\varphi(1)]^2=D(x^n-1)=(-1)^{\frac{(n-1)(n-2)}{2}}n^n.$$

Consequently,

$$D(\varphi)=(-1)^{\frac{(n-1)(n-2)}{2}}n^{n-2}.$$

840. Let $\varphi(x)=x^n+ax^{n-1}+ax^{n-2}+\dots+a$. Then $\varphi(x)(x-1)=x^{n+1}+(a-1)x^n-a$. Hence,

$$(na+1)^2D(\varphi)=(-1)^{\frac{n(n-1)}{2}}a^{n-1}[(n+1)^{n+1}a+n^n(1-a)^{n+1}].$$

Thus,

$$D(\varphi)=(-1)^{\frac{n(n-1)}{2}}a^{n-1}\frac{(n+1)^{n+1}a+n^n(1-a)^{n+1}}{(1+na)^2}.$$

841. Let $f(x)=a_0(x-x_1)(x-x_2)\dots(x-x_n)$,

$$\varphi(x)=b_0(x-y_1)(x-y_2)\dots(x-y_m).$$

Then

$$D(f\varphi)=(a_0b_0)^{2m+2n-2}\prod_{i<k}(x_i-x_k)^2\prod_{i<k}(y_i-y_k)^2\times\prod_{i=1}^{n}\prod_{k=1}^{m}(x_i-y_k)^2=a_0^{2n-2}\prod_{i<k}(x_i-x_k)^2b_0^{2m-2}$$

$$\times\prod_{i<k}(y_i-y_k)^2\left[a_0^mb_0^n\prod_{i=1}^{n}\prod_{k=1}^{m}(x_i-y_k)\right]^2=D(f)D(\varphi)[R(f,\varphi)]^2.$$

842. $X_{p^m}(x^{p^{m-1}}-1)=x^{p^m}-1$. Consequently,

$$D(X_{p^m})D(x^{p^{m-1}}-1)[R(x^{p^{m-1}}-1,X_{p^m})]^2=D(x^{p^m}-1).$$

Substituting the values of the known quantities, we get

$$D(X_{p^m})=p^{mp^m-(m+1)p^{m-1}}(-1)^{\frac{1}{2}p^{m-1}(p-1)}.$$

843. $X_n\prod_{\delta/n}(x^\delta-1)^{\mu\left(\frac{n}{\delta}\right)}=(x^n-1)\prod_{\substack{\delta/n\\ \delta\neq n}}(x^\delta-1)^{\mu\left(\frac{n}{\delta}\right)}.$

Let ε be a root of X_n. Then

$$X_n'(\varepsilon)=n\varepsilon^{n-1}\prod_{\substack{\delta/n\\ \delta\neq n}}(\varepsilon^{\delta}-1)^{\mu\left(\frac{n}{\delta}\right)}.$$

To simplify computations, let us first find the absolute value of the discriminant of X_n:

$$|D(X_n)|=\prod_{\varepsilon}|X_n'(\varepsilon)|=n^{\varphi(n)}\prod_{\substack{\delta/n\\ \delta\neq n}}\prod_{\varepsilon}(1-\varepsilon^{\delta})^{\mu\left(\frac{n}{\delta}\right)}$$

$$=n^{\varphi(n)}\prod_{\substack{\delta/n\\ \delta\neq n}}[X_{\frac{n}{\delta}}(1)]^{\frac{\varphi(n)}{\varphi\left(\frac{n}{\delta}\right)}\mu\left(\frac{n}{\delta}\right)}.$$

Now, $X_{\frac{n}{\delta}}(1)$ differs from 1, provided only that $\frac{n}{\delta}$ is a power of a prime number. On the other hand, $\mu\left(\frac{n}{\delta}\right)$ is different from 0, provided only that $\frac{n}{\delta}$ is not divisible by the square of a prime. Thus, in the latter product we must retain only those factors corresponding to $\frac{n}{\delta}=p_1, p_2, \ldots, p_k$, where $p_1, p_2, \ldots, p_k$ are distinct prime divisors of the number n.

Thus,

$$|D(X_n)|=\frac{n^{\varphi(n)}}{\prod_{p/n}p^{\varphi(n)/p-1}}.$$

Since all roots of X_n are complex, the sign of the discriminant is equal to $(-1)^{\frac{\varphi(n)}{2}}$. Finally,

$$D(X_n)=(-1)^{\frac{\varphi(n)}{2}}\frac{n^{\varphi(n)}}{\prod_{p/n}p^{\varphi(n)/p-1}}.$$

844. $E_n=n!\left(1+\frac{x}{1}+\ldots+\frac{x^n}{n!}\right)$,

$$E_n'=n!\left(1+\frac{x}{1}+\ldots+\frac{x^{n-1}}{(n-1)!}\right).$$

Hence

$$E_n' = E_n - x^n,$$

$$R(E_n, E_n') = \prod_{i=1}^{n} (-x_i)^n = (-1)^n [(-1)^n n!]^n = (n!)^n,$$

$$D(E_n) = (-1)^{\frac{n(n-1)}{2}} (n!)^n.$$

845. It is easy to establish that

$$(nx+n-a) F_n - x(x+1) F_n' + \frac{a(a-1) \ldots (a-n)}{n!} = 0.$$

Let $x_1, x_2, \ldots, x_n$ be roots of F_n. Then

$$F_n'(x_i) = \frac{c}{x_i(x_i+1)} \quad \text{where } c = \frac{a(a-1) \ldots (a-n)}{n!}.$$

Hence

$$R(F_n, F_n') = \frac{c^n}{\Pi x_i \, \Pi(x_i+1)} = \frac{c^n}{\frac{a(a-1) \ldots (a-n+1)}{n!} \cdot \frac{(a-1) \ldots (a-n)}{n!}}$$

$$= \frac{a^{n-1}(a-1)^{n-2}(a-2)^{n-2} \ldots (a-n+1)^{n-2}(a-n)^{n-1}}{(n!)^{n-2}},$$

$$D(F_n) = (-1)^{\frac{n(n-1)}{2}} \frac{a^{n-1}(a-1)^{n-2}(a-2)^{n-2} \ldots (a-n+1)^{n-2}(a-n)^{n-1}}{(n!)^{n-2}}.$$

846. $P_n' = nP_{n-1}$. Hence

$$R(P_n, P_n') = n^n R(P_n, P_{n-1}).$$

Furthermore

$$P_n - xP_{n-1} + (n-1) P_{n-2} = 0.$$

Consequently, $P_n(\xi) = -(n-1) P_{n-2}(\xi)$ if ξ is a root of P_{n-1}, and therefore

$$R(P_n, P_{n-1}) = (-1)^{n-1}(n-1)^{n-1} R(P_{n-2}, P_{n-1})$$
$$= (-1)^{n-1}(n-1)^{n-1} R(P_{n-1}, P_{n-2}).$$

It is now easy to establish that

$$R(P_n, P_{n-1}) = (-1)^{\frac{n(n-1)}{2}} (n-1)^{n-1}(n-2)^{n-2} \ldots 2^2 \cdot 1.$$

Finally

$$D(P_n) = 1 \cdot 2^2 \cdot 3^3 \ldots (n-1)^{n-1} n^n.$$

847. $D(P_n) = 1 \cdot 2^3 \cdot 3^5 \ldots n^{2n-1}$.

848. $D(P_n) = 2^{n-1} n^n$.

849. $D(P_n) = (n+1)^{n-1} \cdot 2^{n(n-1)}$.

850. $D(P_n) = 1 \cdot 2^3 \cdot 3^5 \ldots n^{2n-1} \cdot 1^{2(n-1)} \cdot 3^{2(n-2)} \ldots (2n-3)^2$.

851. $D(P_n) = 2^2 \cdot 3^4 \ldots n^{2n-2} \cdot (n+1)^{n-1}$.

852. Let $f(x) = x^n + a_1 x^{n-1} + \ldots + a_n = (x-x_1)(x-x_2) \ldots (x-x_n)$.

$D(f)=\Pi\,(x_i-x_k)^2$. We seek the maximum of $D(f)$ by the rule for finding a relative maximum by solving the system of equations

$$x_1^2+x_2^2+\ \dots\ +x_n^2=n\,(n-1)\,R^2,$$

$$\frac{\partial}{\partial x_i}\left(D-\lambda\,(x_1^2+x_2^2+\dots+x_n^2)\right)=0.$$

It is easy to see that

$$\frac{\partial D}{\partial x_i}=\frac{Df''(x_i)}{f'(x_i)}\,.$$

We thus have

$$f''(x)\,D-2\lambda\,x_i f'(x_i)=0 \text{ for } i=1,2,\dots,n.$$

Thus, a polynomial $f(x)$ that maximizes the discriminant must satisfy the differential equation

$$cf(x)-2\lambda xf'(x)+Df''(x)=0$$

where c is some constant. Dividing by $\dfrac{c}{n}$ and comparing the coefficients of x^n, we find that the differential equation must have the form

$$nf(x)-xf'(x)+c'f''(x)=0$$

where c' is a new constant.

Comparing the coefficients of x^{n-1} and x^{n-2}, we find $a_1=0$, $a_2=-\dfrac{n\,(n-1)}{2}c'$. Now we can determine c'. Indeed,

$$n\,(n-1)\,R^2=x_1^2+x_2^2+\ \dots\ +x_n^2=a_1^2-2a_2=n\,(n-1)\,c'$$

whence $c'=R^2$.

Continuing to compare the coefficients, we find that $f(x)$ is of the form

$$f(x)=x^n-\frac{n\,(n-1)}{2}\,R^2\,x^{n-2}+\frac{n\,(n-1)\,(n-2)\,(n-3)}{2\cdot 4}\,R^4x^{n-4}-\dots$$

It is easy to see that

$$f(x)=R^n\,P_n\left(\frac{x}{R}\right)$$

where P_n is a Hermite polynomial.

$$D(f)=R^{n\,(n-1)}\cdot 1\cdot 2^2\cdot 3^3\ \dots\ n^n.$$

This is the desired maximum of the discriminant.

853. $2^{2n}\,(-1)^n\,a_0\,a_n\,[D(f)]^2$.

854. $m^{mn}\,(-1)^{\frac{m\,(m-1)\,n}{2}}\,a_0^{m-1}\,a_n^{m-1}\,[D(f)]^m$.

855. $F(x)=\prod\limits_{i=1}^{n}\,(\varphi(x)-x_i)$.

Hence

$$D(F)=\prod_{i=1}^{n}\,D(\varphi(x)-x_i)\left[\prod_{i<k}\,R(\varphi(x)-x_i,\,\varphi(x)-x_k)\right]^2.$$

It is furthermore obvious that

$$R(\varphi(x)-x_i,\ \varphi(x)-x_k)=(x_i-x_k)^m.$$

Therefore

$$D(F)=\prod_{i=1}^{n} D(\varphi(x)-x_i),\ \prod_{i<k}(x_i-x_k)^{2m}=[D(f)]^m\prod_{i=1}^{n} D(\varphi(x)-x_i)$$

which completes the proof.

856. $(y+1)\ (y-5)\ (y-19)=0$.

857. (a) *Solution.* $x^3=3x+4$. Let $y=1+x+x^2$ where x is a root of the given equation. Then

$$yx=x+x^2+x^3=x+x^2+3x+4=4+4x+x^2,$$
$$yx^2=4x+4x^2+x^3=4x+4x^2+3x+4=4+7x+4x^2.$$

Eliminating x, we get

$$\begin{vmatrix} 1-y & 1 & 1 \\ 4 & 4-y & 1 \\ 4 & 7 & 4-y \end{vmatrix}=0,$$

$$y^3-9y^2+9y-9=0;$$

(b) $y^3-7y^2+3y-1=0$;

(c) $y^4+5y^3+9y^2+7y-6=0$;

(d) $y^4-12y^3+43y^2-49y+20=0$.

858. (a) $y^3-2y^2+6y-4=0,\ x=-\dfrac{y^2-2y+4}{2}$;

(b) $y^4-9y^3+31y^2-45y+13=0,\ x=\dfrac{y^2-3y+2}{3}$;

(c) $y^4+2y^3-y^2-2y+1=0$, there is no inverse transformation.

859. $y^3-y^2-2y+1=0$.

The transformed equation coincides with the original one. This means that among the roots of the original equation there are roots x_1 and x_2 connected by the relation $x_2=2-x_1^2$.

860. Let $x_2=\varphi(x_1)$, where $\varphi(x_1)$ is a rational function with rational coefficients. Without loss of generality, we can take it that $x_2=ax_1^2+bx_1+c$. The numbers $ax_1^2+bx_1+c$, $ax_2^2+bx_2+c$, $ax_3^2+bx_3+c$ are roots of a cubic equation with rational coefficients, one of the roots of which coincides with the root $x_2=ax_1^2+bx_1+c$ of the given equation. Since the given equation is irreducible, the other roots must coincide as well. Thus, either $ax_2^2+bx_2+c=x_3$, $ax_3^2+bx_3+c=x_1$, or $ax_2^2+bx_2+c=x_1$, $ax_3^2+bx_3+c=x_3$. The latter equation is impossible since x_3 cannot be a root of a quadratic equation with rational coefficients. Thus, on our assumption the roots of the given equation are connected by the relations

$$x_2 = ax_1^2 + bx_1 + c,$$
$$x_3 = ax_2^2 + bx_2 + c,$$
$$x_1 = ax_3^2 + bx_3 + c.$$

Consequently

$$\sqrt{D} = (x_2 - x_1)(x_3 - x_2)(x_1 - x_3)$$
$$= [ax_1^2 + (b-1)x_1 + c][ax_2^2 + (b-1)x_2 + c][ax_3^2 + (b-1)x_3 + c]$$

is a rational number, being a symmetric function of x_1, x_2, x_3 with rational coefficients. The necessity of the condition is proved.

Now suppose that the discriminant D is the square of the rational number d. Then

$$x_2 - x_3 = \frac{d}{(x_1 - x_2)(x_1 - x_3)} = \frac{d}{3x_1^2 + 2ax_1 + b}.$$

On the other hand,

$$x_2 + x_3 = -a - x_1.$$

From this it follows that x_2 and x_3 are rational functions of x_1. This proves the sufficiency of the condition.

861. (a) $\dfrac{2+\sqrt{2}+\sqrt{6}}{4}$,

(b) $\dfrac{-3+7\sqrt[3]{2}-\sqrt[3]{4}}{23}$,

(c) $1+3\sqrt[4]{2}+2\sqrt{2}-\sqrt[4]{8}$.

862. (a) $\dfrac{\alpha^2-\alpha+1}{3}$, (b) $17\alpha^2-3\alpha+55$, (c) $3-10\alpha+8\alpha^2-3\alpha^3$,

(d) the denominator vanishes for one of the roots of the equation.

863. $mx_1^2+nx_1+p = \dfrac{(pm-bm^2+amn-n^2)x_1+(amp-np-cm^2)}{mx_1+ma-n}$.

864. If

$$x_2 = \frac{\alpha x_1 + \beta}{\gamma x_1 + \delta},$$

then

$$x_3 = \frac{\alpha x_2 + \beta}{\gamma x_2 + \delta}$$

$$x_1 = \frac{\alpha x_3 + \beta}{\gamma x_3 + \delta} = \frac{(\alpha^2+\beta\gamma)x_2 + (\alpha+\delta)\beta}{(\alpha+\delta)\gamma x_2 + (\beta\gamma+\delta^2)}.$$

On the other hand, $x_1 = \dfrac{-\delta x_2+\beta}{\gamma x_2-\alpha}$, whence follows the necessity of the relation

$$\alpha\delta - \beta\gamma = (\alpha+\delta)^2.$$

865. Let

$$a_0x^n+a_1x^{n-1}+\ \dots\ +a_n=a_0(x-x_1)\ (x-x_2)\ \dots(x-x_n).$$

Then

$$a_0x^n-a_1x^{n-1}+\ \dots\ +(-1)^na_n=a_0(x+x_1)\ (x+x_2)\ \dots(x+x_n).$$

Multiplying these equations, we get

$$a_0^2(x^2-x_1^2)(x^2-x_2^2)\ \dots\ (x^2-x_n^2)$$
$$=(a_0x^n+a_2x^{n-2}+\dots)^2-(a_1x^{n-1}+a_3x^{n-3}+\dots)^2.$$

From this we conclude that in order to perform the transformation $y=x^2$, it is necessary to substitute y for x^2 in the equation

$$(a_0x^n+a_2x^{n-2}+\dots)^2-(a_1x^{n-1}+a_3x^{n-3}+\dots)^2=0.$$

866. The desired equation results from a substitution of y for x^3 in the equation

$$(a_0x^n+a_3x^{n-3}+\dots)^3+(a_1x^{n-1}+a_4x^{n-4}+\dots)^3$$
$$+(a_2x^{n-2}+a_5x^{n-5}+\dots)^3-3(a_0x^n+a_3x^{n-3}+\dots)$$
$$\times(a_1x^{n-1}+a_4x^{n-4}+\dots)(a_2x^{n-2}+a_5x^{n-5}+\dots)=0.$$

867. There are only a finite number of polynomials $x^n+a_1x^{n-1}+\dots$ with integral coefficients the moduli of whose roots do not exceed 1, because the coefficients of such polynomials are obviously restricted:

$$|a_k|\leqslant\frac{n(n-1)\ \dots\ (n-k+1)}{k!}.$$

Let $f=x^n+a_1x^{n-1}+\ \dots\ +a_n$, $a_n\neq0$ be one of such polynomials and let $x_1, x_2, \dots, x_n$ be its roots. Denote $f_m=(x-x_1^m)\ \ (x-x_2^m)\dots(x-x_n^m)$. All polynomials f_m have integral coefficients and all their roots do not exceed 1 in absolute value. Hence, there are only a finite number of distinct roots among them. Choose an infinite sequence of integers $m_0<m_1<m_2<\dots$ such that $f_{m_0}=f_{m_1}=f_{m_2}=\ \dots$. This signifies that

$$x_1^{m_i}=x_{\alpha_1}^{m_0},$$
$$x_2^{m_i}=x_{\alpha_2}^{m_0},$$
$$\dots\dots$$
$$x_n^{m_i}=x_{\alpha_n}^{m_0}$$

where $(\alpha_1, \alpha_2, \dots, \alpha_n)$ is some permutation of the indices $1, 2, \dots, n$. Since there are infinitely many exponents m_i and only a finite number of permutations, there will be two (and infinitely many) exponents m_{i_1} and m_{i_2} to which corresponds one and the same permutation $(\alpha_1, \alpha_2, \dots, \alpha_n)$. For such exponents we have the equations

$$x_1^{m_{i_1}}=x_1^{m_{i_2}},$$
$$x_2^{m_{i_1}}=x_2^{m_{i_2}},$$
$$\dots\dots\dots$$
$$x_n^{m_{i_1}}=x_n^{m_{i_2}}$$

which show that $x_1, x_2, \ldots, x_n$ are roots of unity of degree $m_{i_2}-m_{i_1}$ because $x_1, x_2, \ldots, x_n$ are nonzero, by virtue of the condition $a_n \neq 0$.

868. Let $F(x_1, x_2, \ldots, x_n)$ be a polynomial which changes sign under odd permutations of the variables. Since $F(x_1, x_2, x_3, \ldots, x_n) = -F(x_1, x_2, \ldots, x_n) = 0$, $F(x_1, x_2, \ldots, x_n)$ is divisible by $x_1 - x_2$. In similar fashion we prove that $F(x_1, x_2, \ldots, x_n)$ is divisible by all the differences $x_i - x_k$. Hence, $F(x_1, x_2, \ldots, x_n)$ is divisible by $\Delta = \prod_{i>k} (x_i - x_k)$ equal to the Vandermonde determinant. Since the determinant Δ changes sign under odd permutations of the variables, $\frac{F}{\Delta}$ is a symmetric polynomial.

869. Let $\varphi(x_1, x_2, \ldots, x_n)$ be a polynomial that does not change under even permutations of the variables. Denote by $\bar{\varphi}(x_1, x_2, \ldots, x_n)$ a polynomial obtained from $\varphi(x_1, x_2, \ldots, x_n)$ by means of some definite odd permutation.

It is easy to verify that for every odd permutation, φ goes into $\bar{\varphi}$, $\bar{\varphi}$ goes into φ. Hence, $\varphi + \bar{\varphi}$ does not change under all permutations, $\varphi - \bar{\varphi}$ changes sign under odd permutations.

Next,

$$\varphi = \frac{\varphi + \bar{\varphi}}{2} + \frac{\varphi - \bar{\varphi}}{2} = F_1 + F_2 \Delta$$

where Δ is the Vandermonde determinant. On the basis of the result of Problem 868, F_2 is a symmetric polynomial; F_1 is also a symmetric polynomial since it does not change under all permutations of the variables.

870. $(f_1^2 - f_2)\Delta$, where f_1, f_2 are elementary symmetric polynomials in $x_1, x_2, \ldots, x_n$.

871. $$u^3 + a(\alpha+\beta+\gamma)u^2 + [(\alpha^2+\beta^2+\gamma^2)b + (\alpha\beta+\alpha\gamma+\beta\gamma)(a^2-b)]u$$
$$+ c(\alpha^3+\beta^3+\gamma^3) + \frac{ab-3c}{2}(\alpha^2\beta+\alpha\beta^2+\alpha^2\gamma+\alpha\gamma^2+\beta^2\gamma+\beta\gamma^2)$$
$$+ \alpha\beta\gamma(a^3-3ab+6c) + \frac{(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)}{2}\sqrt{\Delta} = 0$$

where Δ is the discriminant of the given equation.

872. $u^3 - 3pp'u - \dfrac{27qq' + \sqrt{\Delta\Delta'}}{2} = 0$, where Δ and Δ' are discriminants of the given equations.

873. Let $y = ax^2 + bx + c$ be the Tschirnhausen transformation connecting the given equations. Then, for some choice of numbering,

$$x_1 y_1 + x_2 y_2 + x_3 y_3 = a(x_1^3 + x_2^3 + x_3^3) + b(x_1^2 + x_2^2 + x_3^2) + c(x_1 + x_2 + x_3)$$

will be a rational number. Hence, one of the equations

$$u^3 - 3pp'u - \frac{27qq' \pm \sqrt{\Delta\Delta'}}{2} = 0$$

(Problem 872) has a rational root. Whence it follows that $\sqrt{\Delta\Delta'}$ will be a rational number. This proves the necessity of the condition.

Conversely, let the equation

$$u^3 - 3pp'u - \frac{27qq' \pm \sqrt{\Delta\Delta'}}{2} = 0 \qquad (*)$$

have a rational root u.

It is easy to see that the discriminant of equation (*) is equal to $\frac{27^2}{4}(q\sqrt{\Delta'}-\sqrt{\Delta}\,q')^2$ and, consequently, differs from $\sqrt{\Delta}$ by a factor equal to the square of a rational number. Hence, the difference $u'-u''$ of the second and third roots of the equation differ by a rational factor from $\sqrt{\Delta}$.

For y_1, y_2, y_3 we have the system of equations:

$$y_1+y_2+y_3=0,$$
$$x_1 y_1+x_2 y_2+x_3 y_3=u,$$
$$(x_2-x_3)\,y_1+(x_3-x_1)\,y_2+(x_1-x_2)\,y_3=u'-u''=r\sqrt{\Delta}.$$

From this system we find

$$y_1=\frac{-3ux_1+(x_2-x_3)\,r\sqrt{\Delta}}{6p}.$$

But $(x_2-x_3)\sqrt{\Delta}$ is expressed rationally in terms of x_1. This proves the sufficiency of the conditions.

874. The variables $x_1, x_2, \ldots, x_n$ are expressible linearly in terms of f_1, $\eta_1, \eta_2, \ldots, \eta_{n-1}$. Hence, every polynomial in $x_1, x_2, \ldots, x_n$ may be represented as a polynomial in $f_1, \eta_1, \eta_2, \ldots, \eta_{n-1}$:

$$F(x_1, x_2, \ldots, x_n)=\sum Af_1^{\alpha_0}\eta_1^{\alpha_1}\eta_2^{\alpha_2}\ldots\eta_{n-1}^{\alpha_{n-1}}.$$

In a circular permutation of the variables $x_1, x_2, \ldots, x_n$, the monomial $Af_1^{\alpha_0}\eta_1^{\alpha_1}\eta_2^{\alpha_2}\ldots\eta_{n-1}^{\alpha_{n-1}}$ acquires the factor $\varepsilon^{-(\alpha_1+2\alpha_2+\ldots+(n-1)\alpha_{n-1})}$. Hence, so that $F(x_1, x_2, \ldots, x_n)$ should not change in circular permutations of the variables, it is necessary and sufficient that $\alpha_1+2\alpha_2+\ldots+(n-1)\alpha_{n-1}$ be divisible by n.

875. We can take $f_1, \eta_1^n, \eta_2\eta_1^{-2}, \ldots, \eta_{n-1}\eta_1^{-(n-1)}$.

876. Let $\eta_1=x_1+x_2\varepsilon+x_3\varepsilon^2$, $\eta_2=x_1+x_2\varepsilon^2+x_3\varepsilon$ where $\varepsilon=-\frac{1}{2}+i\frac{\sqrt{3}}{2}$. Then $\frac{\eta_1^2}{\eta_2}=\varphi_1+i\sqrt{3}\,\varphi_2$ where φ_1 and φ_2 are some rational functions of x_1, x_2, x_3 with rational coefficients that do not change under a circular permutation of x_1, x_2, x_3. It is easy to see that every rational function of x_1, x_2, x_3 that remains fixed under circular permutations of the variables is expressible rationally in terms of $f_1=x_1+x_2+x_3$, φ_1 and φ_2.

It is sufficient to prove this for $\eta_2\eta_1^{-2}$ and η_1^3. But

$$\eta_2\eta_1^{-2}=\frac{1}{\varphi_1+i\varphi_2\sqrt{3}},$$

$$\eta_1=\left(\frac{\eta_1^2}{\eta_2}\right)^2\cdot\frac{\eta^2}{\eta_1}=(\varphi_1+i\varphi_2\sqrt{3})^2(\varphi_1-i\varphi_2\sqrt{3}).$$

877. For $n=4$,

$$\eta_1=x_1+ix_2-x_3-ix_4,$$
$$\eta_2=x_1-x_2+x_3-x_4,$$
$$\eta_3=x_1-ix_2-x_3+ix_4.$$

Put $\theta_1=\eta_1\eta_3$, $\theta_2+i\theta_3=\frac{\eta_1\eta_2}{\eta_3}$, $\theta_2-i\theta_3=\frac{\eta_3\eta_2}{\eta_1}$. θ_1; θ_2, θ_3 are rational functions (with rational coefficients) of x_1, x_2, x_3, x_4 that do not change under circular permutations. It is easy to see that together with $f=x_1+x_2+x_3+x_4$ they form a system of elementary functions. Indeed,

$$\eta_2\eta_1^{-2}=\frac{\theta_2-i\theta_3}{\theta_1},\quad \eta_3\eta_1^{-3}=\frac{\theta_2-i\theta_3}{\theta_1(\theta_2+i\theta_3)},$$

$$\eta_1^4=\frac{\theta_1^4(\theta_2+i\theta_3)}{\theta_2-i\theta_3}.$$

878. Let
$$\eta_1=x_1\varepsilon+x_2\varepsilon^2+x_3\varepsilon^3+x_4\varepsilon^4+x_5,$$
$$\eta_2=x_1\varepsilon^2+x_2\varepsilon^4+x_3\varepsilon+x_4\varepsilon^3+x^5,$$
$$\eta_3=x_1\varepsilon^3+x_2\varepsilon+x_3\varepsilon^4+x_4\varepsilon^2+x_5,$$
$$\eta_4=x_1\varepsilon^4+x_2\varepsilon^3+x_3\varepsilon^2+x_4\varepsilon+x_5.$$

Let us consider the rational function $\lambda_1=\frac{\eta_1\eta_2}{\eta_3}$ and arrange it in powers of ε, replacing 1 by $-\varepsilon-\varepsilon^2-\varepsilon^3-\varepsilon^4$:

$$\lambda_1=\varepsilon\varphi_1+\varepsilon^2\varphi_2+\varepsilon^3\varphi_3+\varepsilon^4\varphi_4.$$

The coefficients of φ_1, φ_2, φ_3, φ_4 are rational numbers. Substituting ε^2, ε^3 and ε^4 for ε, we get

$$\lambda_2=\frac{\eta_2\eta_4}{\eta_1}=\varepsilon^2\varphi_1+\varepsilon^4\varphi_2+\varepsilon\varphi_3+\varepsilon^3\varphi_4,$$

$$\lambda_3=\frac{\eta_3\eta_1}{\eta_4}=\varepsilon^3\varphi_1+\varepsilon\varphi_2+\varepsilon^4\varphi_3+\varepsilon^2\varphi_4,$$

$$\lambda_4=\frac{\eta_4\eta_3}{\eta_2}=\varepsilon^4\varphi_1+\varepsilon^3\varphi_4+\varepsilon^2\varphi_3+\varepsilon\varphi_4.$$

For the "elementary functions" we can take f, φ_1, φ_2, φ_3, φ_4. Indeed λ_1, λ_2, λ_3, λ_4 can be expressed rationally in terms of them. Furthermore,

$$\eta_2\eta_1^{-2}=\lambda_1^{-1}\lambda_2\lambda_4^{-1},\quad \eta_4\eta_1^{-4}=\lambda_1^{-2}\lambda_2\lambda_3^{-1}\lambda_4^{-1},$$
$$\eta_3\eta_1^{-3}=\lambda_1^{-2}\lambda_2\lambda_4^{-1},\qquad \eta_1^5=\lambda_1^3\lambda_2^{-1}\lambda_3\lambda_4^2.$$

CHAPTER 7

LINEAR ALGEBRA

879. (a) Dimensionality $r=2$, the basis is generated, for example, by X_1 and X_2;

(b) $r=2$, the basis is generated, say, by X_1, and X_2;

(c) $r=2$, the basis is generated, say, by X_1 and X_2.

880. (a) The dimensionality of the intersection is equal to one, the basis vector

$$Z=(5,\ -2,\ -3,\ -4)=X_1-4X_2=3Y_1-Y_2.$$

The dimensionality of the sum is equal to 3, the basis is generated, say, by the vectors Z, X_1, Y_1.

(b) The sum coincides with the first space, the intersection with the second.

(c) The sum is the entire four-dimensional space, the intersection consists only of the zero vector.

881. (a) $\left(\frac{5}{4}, \frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}\right)$, (b) $(1, 0, -1, 0)$,

882. (a) $x_1' = \frac{x_1+x_2-x_3-x_4}{2}$, $x_2' = \frac{x_1-x_2+x_3-x_4}{2}$,

$$x_3' = \frac{x_1-x_2-x_3+x_4}{2}, \quad x_4' = \frac{-x_1+x_2+x_3+x_4}{2};$$

(b) $x_1' = x_2 - x_3 + x_4$, $x_2' = -x_1 + x_2$, $x_3' = x_4$, $x_4' = x_1 - x_2 + x_3 - x_4$.

883. $x_1'x_2' + x_3'x_4' = \frac{1}{8}$.

884. Let $a_0 + a_1 \cos x + a_2 \cos^2 x + \ldots + a_n \cos^n x = b_0 + b_1 \cos x + b_2 \cos 2x + \ldots + b_n \cos nx$.

Then $a_0 = b_0 - b_2 + b_4 - \ldots$,

$$a_k = 2^{k-1}\left[b_k + \sum_{1 \leqslant p \leqslant \frac{n-k}{2}} (-1)^p \frac{(k+2p)(k+p-1)(k+p-2)\ldots(k+1)}{p!} b_{k+2p}\right],$$

$$b_0 = a_0 + \sum_{1 \leqslant p \leqslant \frac{n}{2}} 2^{-2p} C_{2p}^p a_{2p},$$

$$b_k = 2^{1-k}\left(a_k + \sum_{1 \leqslant p \leqslant \frac{n-k}{2}} 2^{-2p} C_{k+2p}^p a_{k+2p}\right).$$

885. The point of intersection with the first line has the coordinates $\left(\frac{14}{3}, \frac{1}{9}, \frac{7}{9}, \frac{11}{9}\right)$, with the second, the coordinates $(42, 1, 7, 11)$.

886. The straight lines $X_0 + tX_1$, $Y_0 + tY_1$ lie in the manifold $X_0 + t(Y_0 - X_0) + t_1X_1 + t_2Y_1$.

887. For the problem to be solvable for the straight lines $X_0 + tX_1$, $Y_0 + tY_1$, it is necessary and sufficient that the vectors X_0, Y_0, X_1, Y_1 be linearly dependent. This is equivalent to being able to embed straight lines in a three-dimensional subspace containing the coordinate origin.

888. The planes $X_0 + t_1X_1 + t_2X_2$ and $Y_0 + t_1Y_1 + t_2Y_2$ can be embedded in the manifold $X_0 + t(Y_0 - X_0) + t_1X_1 + t_2X_2 + t_3Y_1 + t_4Y_2$.

889. There are 6 such cases:

(1) the planes have no common points and cannot be embedded in a four-dimensional linear manifold (the planes cross absolutely);

(2) the planes have no common points, are contained in a four-dimensional manifold, but are not embedded in a three-dimensional manifold (they cross parallel to a straight line);

(3) the planes have no common points and are embedded in a three-dimensional manifold (the planes are parallel);

(4) the planes have one common point. In this case, they are embedded in a four-dimensional manifold, but cannot be embedded in a three-dimensional manifold;

(5) the planes intersect along a straight line;

(6) the planes are coincident.

In three-dimensional space, only cases 3, 5, 6 are realized.

890. Let $Q=X_0+P$ be a linear manifold, let P be a linear space. If $X_1 \in Q$ and $X_2 \in Q$, then $X_1=X_0+Y_1$, $X_2=X_0+Y_2$, where Y_1 and Y_2 belong to P. Then $\alpha X_1+(1-\alpha)X_2=X_0+\alpha Y_1+(1-\alpha)Y_2 \in Q$ for any α. Conversely, let Q be a set of vectors containing, together with the vectors X_1, X_2, their linear combination $\alpha X_1+(1-\alpha)X_2$ for arbitrary α. Let X_0 be some fixed vector from Q and let P denote the set of all vectors $Y=X-X_0$. If $Y \in P$, then $cY \in P$ for any c, because $cY=cX+(1+c)X_0-X_0$. Furthermore, if $Y_1=X_1-X_0 \in P$ and $Y_2=X_2-X_0 \in P$, then $\alpha Y_1+(1-\alpha)Y_2=\alpha X_1+(1-\alpha)X_2-X_0 \in P$ for any α. Now let us take some fixed α, $\alpha \neq 0$, $\alpha \neq 1$, arbitrary c_1, c_2. Then $\frac{c_1}{\alpha} Y_1 \in P$, $\frac{c_2}{1-\alpha} Y_2 \in P$ for any $Y_1, Y_2 \in P$, and, hence, also

$$c_1 Y_1+c_2 Y_2=\alpha \frac{c_1}{\alpha} Y_1+(1-\alpha)\frac{c_2}{1-\alpha} Y_2 \in P.$$

Consequently, P is a linear space and Q is a linear manifold.

Remark. The result is not true if the base field is a field of residues modulo 2.

891. (a) 9, (b) 0.

892. (a) 90°, (b) 45°, (c) $\cos \varphi=\frac{3}{\sqrt{77}}$.

893. $\cos \varphi=\frac{1}{\sqrt{n}}$.

894. $\cos A=\frac{5}{\sqrt{39}}$, $\cos B=\frac{8}{\sqrt{78}}$, $\cos C=-\frac{\sqrt{2}}{3}$.

895. $\sqrt{n}$.

896. For odd n there are no orthogonal diagonals. For $n=2m$, the number of diagonals orthogonal to the given one is equal to C_{2m-1}^{m-1}.

897. The coordinates of the points are represented by the rows of the matrix

$$\left\|\begin{array}{ccccc} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \sqrt{\frac{3}{4}} & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{\sqrt{12}} & \sqrt{\frac{4}{6}} & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{1}{2} & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{24}} & \frac{1}{\sqrt{2n(n-1)}} & \sqrt{\frac{n+1}{2n}} \end{array}\right\|.$$

898. $R=\sqrt{\frac{n}{2(n+1)}}$.

The coordinates of the centre are

$$\left(\frac{1}{2},\ \frac{1}{\sqrt{12}},\ \ldots,\ \frac{1}{\sqrt{2n(n-1)}},\ \frac{1}{\sqrt{2(n+1)n}}\right).$$

899. $\left(\frac{3}{\sqrt{15}},\ \frac{1}{\sqrt{15}},\ \frac{2}{\sqrt{15}},\ \frac{1}{\sqrt{15}}\right)$.

900. $\left(0,\ \frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}},\ 0\right)$.

901. For the other two vectors we can take, say,

$\frac{1}{\sqrt{26}}(0,\ -4,\ 3,\ 1)$ and $\frac{1}{3\sqrt{26}}(-13,\ 5,\ 6,\ 2)$.

902. $(1, 2, 1, 3)$, $(10, -1, 1, -3)$, $(19, -87, -61, 72)$.

903. For example, $\begin{pmatrix} 0 & 7 & 3 & -4 & -2 \\ 39 & -37 & 51 & -29 & 5 \end{pmatrix}$.

904. The system is interpreted as a problem in seeking vectors orthogonal to a system of vectors depicting the coefficients of the equations. The set of sought-for vectors is a space orthogonally complementary to the space generated by the given vectors. The fundamental system of solutions is the basis of the space of the desired vectors.

905. For example, $\frac{1}{\sqrt{6}}(1,\ 0,\ 2,\ -1)$, $\frac{1}{\sqrt{498}}(1,\ 12,\ 8,\ 17)$.

906. (a) $X'=(3, 1, -1, -2) \in P$, (b) $X'=(1, 7, 3, 3) \in P$,

$X''=(2, 1, -1, 4) \perp P$, $X''=(-4, -2, 6, 0) \perp P$.

907. Let $A_1, A_2, \ldots, A_m$ be linearly independent, and let P be the space spanned by them. Furthermore, let $X=Y+Z$, $Y \in P$, $Z \perp P$.

Set $$Y=c_1A_1+c_2A_2+\ \ldots\ +c_mA_m.$$

Form a system of equations to determine $c_1, c_2, \ldots, c_m$; to do this, use scalar multiplication of the latter equation by A_i, $i=1, 2, \ldots, m$ and take into account that $YA_i=XA_i$.

We obtain

$$\begin{aligned} c_1A_1^2+c_2A_1A_2+\ \cdots\ +c_mA_1A_m &= XA_1, \\ c_1A_2A_1+c_2A_2^2+\ \cdots\ +c_mA_2A_m &= XA_2, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ c_1A_mA_1+c_2A_mA_2+\ \cdots\ +c_mA_m^2 &= XA_m. \end{aligned}$$

The determinant Δ of this system is nonzero by virtue of the linear independence of $A_1, A_2, \ldots, A_m$.

We find $c_1, c_2, \ldots, c_m$ and substitute them into the expression for Y. This yields

$$Y=\frac{1}{\Delta}\begin{vmatrix} 0 & -A_1 & -A_2 & \cdots & -A_m \\ XA_1 & A_1^2 & A_1A_2 & \cdots & A_1A_m \\ XA_2 & A_2A_1 & A_2^2 & \cdots & A_2A_m \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ XA_m & A_mA_1 & A_mA_2 & \cdots & A_m^2 \end{vmatrix}$$

and

$$Z=\frac{1}{\Delta}\begin{vmatrix} X & A_1 & A_2 & \cdots & A_m \\ A_1X & A_1^2 & A_1A_2 & \cdots & A_1A_m \\ A_2X & A_2A_1 & A_2^2 & \cdots & A_2A_m \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ A_mX & A_mA_1 & A_mA_2 & \cdots & A_m^2 \end{vmatrix}.$$

These equations are to be understood in the sense that the vectors Y and Z are linear combinations of the vectors in the first row with coefficients equal to the corresponding cofactors.

From this we finally get

$$Y^2=Y(X-Z)=YX=\frac{1}{\Delta}\begin{vmatrix} 0 & -XA_1 & -XA_2 & \cdots & -XA_m \\ A_1X & A_1^2 & A_1A_2 & \cdots & A_1A_m \\ A_2X & A_2A_1 & A_2^2 & \cdots & A_2A_m \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ A_mX & A_mA_1 & A_mA_2 & \cdots & A_m^2 \end{vmatrix}$$

and

$$Z^2=Z(X-Y)=ZX=\frac{1}{\Delta}\begin{vmatrix} X^2 & XA_1 & XA_2 & & XA_m \\ A_1X & A_1^2 & A_1A_2 & \cdots & A_1A_m \\ A_2X & A_2A_1 & A_2^2 & \cdots & A_2A_m \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ A_mX & A_mA_1 & A_mA_2 & \cdots & A_m^2 \end{vmatrix}.$$

908. Let Y be some vector of the space P and let X' be the orthogonal projection of the vector X on P.

Then

$$\cos(X, Y)=\frac{XY}{|X|\cdot|Y|}=\frac{X'Y}{|X|\cdot|Y|}=\frac{|X'|\cdot|Y|\cdot\cos(X', Y)}{|X|\cdot|Y|}$$

$$=\frac{|X'|}{|X|}\cos(X' Y)$$

whence it follows that $\cos(X, Y)$ attains a maximum value for those Y for which $\cos(X', Y)=1$, that is, for $Y=\alpha X'$ when $\alpha>0$.

909. (a) 45°, (b) 90°.

910. $\sqrt{\frac{m}{n}}$.

911. $|X-Y|^2=|(X-X')+(X'-Y)|^2=|X-X'|^2+|X'-Y|^2\geqslant$ $\geqslant|X-X'|^2$. The equation is possible only for $Y=X'$.

912. (a) $\sqrt{7}$, (b) $\sqrt{\frac{2}{3}}$.

913. $\frac{n!}{(n+1)(n+2)\ \dots\ 2n\sqrt{2n+1}}$.

914. The required shortest distance is equal to the shortest distance from point X_0-Y_0 to the space $P+Q$.

915. Let one of the vertices lie at the origin of coordinates and let X_1, X_2, ..., X_n be vectors emanating from the origin to the other vertices. It is easy to see that $X_i^2=1$, $X_iX_j=\frac{1}{2}$. The manifold passing through the first $m+1$ vertices is a space $t_1X_1+\dots+t_mX_m$. The manifold passing through the other $n-m$ vertices is $X_n+t_{m+1}(X_{m+1}-X_n)+\dots+t_{n-1}(X_{n-1}-X_n)$. The desired shortest distance is the distance from X_n to the space P generated by the vectors $X_1, X_2, \dots, X_m, X_n-X_{m+1}, \dots, X_n-X_{n-1}$.

Let

$$X_n=t_1X_1+\dots+t_mX_m+t_{m+1}(X_n-X_{m+1})+\dots+t_{n-1}(X_n-X_{n-1})+Y$$

where $Y\perp P$. Forming the scalar product X_n by $X_1, \dots, X_m, X_n-X_{m+1}, \dots, X_n-X_{n-1}$, we get the following system of equations for determining $t_1, \dots, t_{n-1}$:

$$\begin{array}{l}
t_1+\frac{1}{2}t_2+\ \cdots\ +\frac{1}{2}t_m=\frac{1}{2},\quad t_{m+1}+\frac{1}{2}t_{m+2}+\qquad +\frac{1}{2}t_{n-1}=\frac{1}{2},\\
\frac{1}{2}t_1+t_2+\ \cdots\ +\frac{1}{2}t_m=\frac{1}{2},\quad \frac{1}{2}t_{m+1}+t_{m+2}+\ \cdots\ +\frac{1}{2}t_{n-1}=\frac{1}{2},\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
\frac{1}{2}t_1+\frac{1}{2}t_2+\ \cdots\ +t_m=\frac{1}{2},\quad \frac{1}{2}t_{m+1}+\frac{1}{2}t_{m+2}+\ \cdots\ +t_{n-1}=\frac{1}{2},
\end{array}$$

whence $t_1=t_2=\ \cdots\ =t_m=\frac{1}{m+1}$, $t_{m+1}=t_{m+2}=\ \cdots\ =t_{n-1}=\frac{1}{n-m}$.

Consequently

$$Y=\frac{X_{m+1}+X_{m+2}+\ \cdots\ +X_n}{n-m}-\frac{X_1+X_2+\ \cdots\ +X_m}{m+1}.$$

Thus, the common perpendicular is a vector connecting the centres of the chosen faces. The shortest distance is equal to the length of this vector

$$|Y|=\sqrt{\frac{n+1}{2(n-m)(m+1)}}.$$

916. (a) The projection of the vector $(t_1+2t_2,\ t_1-2t_2,\ t_1+5t_2,\ t_1+2t_2)$ on the first plane is $(t_1+2t_2,\ t_1-2t_2,\ 0,\ 0)$. Hence,

$$\cos^2\varphi=\frac{2t_1^2+8t_2^2}{4t_1^2+14t_1t_2+37t_2^2}=\frac{2\lambda^2+8}{4\lambda^2+14\lambda+37},$$

where $\lambda=\frac{t_1}{t_2}$. This expression attains a maximum equal to $\frac{8}{9}$ for $\lambda=-4$.

(b) The angle between any vector of the second plane and its orthogonal projection on the first plane remains fixed and is equal to $\frac{\pi}{4}$.

917. The cube is a set of points whose coordinates satisfy the inequalities $-\frac{a}{2}\leqslant x_i\leqslant\frac{a}{2}$, $i=1,\ 2,\ 3,\ 4$. Here, a is the length of a side of the cube. We pass to new axes, taking for the coordinate vectors, $e_1'=\left(\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}\right)$, $e_2'=\left(\frac{1}{2},\ \frac{1}{2},\ -\frac{1}{2},\ -\frac{1}{2}\right)$, $e_3'=\left(\frac{1}{2},\ -\frac{1}{2},\ \frac{1}{2},\ -\frac{1}{2}\right)$ and $e_4'=$
$$=\left(\frac{1}{2},\ -\frac{1}{2},\ -\frac{1}{2},\ \frac{1}{2}\right).$$

These vectors are orthonormal and their directions coincide with the directions of certain diagonals of the cube. The coordinates of points of the cube in these axes satisfy the inequalities

$$-a\leqslant x_1'+x_2'+x_3'+x_4'\leqslant a,\qquad -a\leqslant x_1'+x_2'-x_3'-x_4'\leqslant a,$$
$$-a\leqslant x_1'-x_2'+x_3'-x_4'\leqslant a,\qquad -a\leqslant x_1'-x_2'-x_3'+x_4'\leqslant a.$$

We get the intersection that interests us by setting $x_1'=0$. It is a solid located in the space spanned by e_2', e_3', e_4', and the coordinates of the points of which satisfy the inequalities $\pm x_2'\pm x_3'\pm x_4'\leqslant a$.

This is a regular octahedron bounded by planes intercepting segments of length a on the axes.

918. $$V^2[B_1,B_2,\dots,B_m]=\begin{vmatrix} B_1^2 & B_1B_2 & \dots & B_1B_m \\ B_2B_1 & B_2^2 & \dots & B_2B_m \\ \dots & \dots & \dots & \dots \\ B_mB_1 & B_mB_2 & \dots & B_m^2 \end{vmatrix}.$$

This formula is readily established by induction if we take into account the result of Problem 907. From this formula it follows immediately that the volume does not depend on the numbering of the vertices and that

$$V[cB_1,B_2,\dots,B_m]=|c|\cdot V[B_1,B_2,\dots,B_m].$$

Now let $B_1=B_1'+B_1''$, C_1, C_1', C_1'' be the orthogonal projections of the vectors B_1, B_1' and B_1'' on the space that is orthogonally complementary to $(B_2,\dots,B_m)$. It is obvious that $C_1=C_1'+C_1''$. By definition,
$V[B_1,B_2,\dots,B_m]=|C_1|\cdot V[B_2,\dots,B_m]$, $V[B_1',B_2,\dots,B_m]=$
$=|C_1'|\cdot V[B_2,\dots,B_m]$, $V[B_1'',B_2,\dots,B_m]=|C_1''|\cdot V[B_2,\dots,B_m]$.
Since $|C_1|\leqslant|C_1'|+|C_1''|$, it follows that $V[B_1,B_2,\dots,B_m]\leqslant V[B_1',B_2,\dots,B_m]+V[B'',B_2,\dots,B_m]$. The equality sign is only possible if C_1' and C_1'' are collinear and in the same direction, which, in turn, occurs if and only if B_1', B_1'' lie in a space spanned by $B_1,B_2,\dots,B_m$, and the coefficients of B_1 in the expressions of B_1', B_1' in terms of $B_1,B_2,\dots,B_m$ have the same signs;

that is, B_1', B_1'' lie "on one side" of the space $(B_2, \ldots, B_m)$ in the space $(B_1, B_2, \ldots, B_m)$.

919. $$V^2[B_1, B_2, \ldots, B_n] = \begin{vmatrix} B_1^2 & B_1B_2 & \ldots & B_1B_n \\ B_2B_1 & B_2^2 & \ldots & B_2B_n \\ \cdot & \cdot & \cdot & \cdot \\ B_nB_1 & B_nB_2 & \ldots & B_n^2 \end{vmatrix} = |\bar{B}B| = |B|^2.$$

where B is a matrix whose columns are the coordinates of the vectors $B_1, B_2, \ldots, B_n$.

920. The following two properties of volume are an immediate consequence of the definition:

(d) $V[B_1 + X_1 B_2, \ldots, B_m] = V[B_1, B_2, \ldots, B_m]$

for any X belonging to the space $(B_2, \ldots, B_m)$ because the points B_1, $B_1 + X$ are at the same distances from $(B_2, \ldots, B_m)$.

(e) $V[B_1, B_2, \ldots, B_m] \leqslant |B_1| \cdot V[B_2, \ldots, B_m]$.

This follows from the fact that the "height", that is, the length of the component of vector B_1 orthogonal to $(B_2, \ldots, B_m)$ does not exceed the length of the vector B_1 itself.

Now let $C_1, C_2, \ldots, C_m$ be orthogonal projections of the vectors $B_1, B_2, \ldots, B_m$ on the space P. Assume that the inequality $V[C_2, \ldots, C_m] \leqslant V[B_2, \ldots, B_m]$ has already been proved. Denote by B_1' the component, orthogonal to $(B_2, \ldots, B_m)$, of the vector B_1, by C_1' its projection on P. Since $B_1' - B_1 \in (B_2, \ldots, B_m)$, we conclude that $C_1' - C \in (C_2, \ldots, C_m)$ and, hence, that $V[C_1, C_2, \ldots, C_m] = V[C_1', C_2, \ldots, C_m] \leqslant |C_1'| \cdot V[C_2, \ldots, C_m]$. But obviously, $C_1| \leqslant |B_1'|$ and, by the induction hypothesis, $V[C_2, \ldots, C_m] \leqslant V[B_2, \ldots, B_m]$. Consequently, $V[C_1, C_2, \ldots, C_m] \leqslant |B_1'| \cdot V[B_2, \ldots, B_m] = V[B_1, B_2, \ldots, B_m]$. There is a basis for induction, since the theorem is obvious for one-dimensional parallelepipeds.

921. From the formula for computing the square of a volume it follows that $V[A_1, \ldots, A_m, B_1, \ldots, B_k] = V[A_1, \ldots, A_m] \cdot V[B_1, \ldots, B_k]$ if each vector A_i is orthogonal to every vector B_j. In the general case, we replace the vectors $B_1, \ldots, B_k$ by their projections $C_1, \ldots, C_k$ on the space that is orthogonally complementary to $(A_1, \ldots, A_m)$. By virtue of the result of the preceding problem, $V[C_1, \ldots, C_k] \leqslant V[B_1, \ldots, B_k]$, whence

$$V[A_1, \ldots, A_m, B_1, \ldots, B_k] = V[A_1, \ldots, A_m, C_1, \ldots, C_k]$$
$$= V[A_1, \ldots, A_m] \cdot V[C_1, \ldots, C_k] \leqslant V[A_1, \ldots, A_m] \cdot V[B_1, \ldots, B_k].$$

The content of this problem coincides with that of Problem 518.

922. This follows immediately from the inequality $V[A_1, \ldots, A_m] \leqslant |A_1| \times \times V[A_2, \ldots, A_m]$, which, in turn, is an immediate consequence of the definition of a volume.

This problem coincides, in content, with Problem 519.

923. A similarity transformation of a solid in n-dimensional space implies a change in volume proportional to the nth power of the expansion factor. For a parallelepiped, this follows immediately from the volume formula, for any other solid, the volume is the limit of the sum of volumes of the parallelepipeds. Hence, the volume $V_n(R)$ of an n-dimensional sphere of radius R is equal to $V_n(1) R^n$.

To compute $V_n(1)$, partition the sphere by a system of parallel $(n-1)$-dimensional "planes" and take advantage of Cavalieri's principle.

Let x be the distance of the cutting "plane" from the centre. The section is an $(n-1)$-dimensional sphere of radius $\sqrt{1-x^2}$

Consequently,

$$V_n(1)=2\int_0^1 V_{n-1}(\sqrt{1-x^2})\,dx=2V_{n-1}(1)\int_0^1(1-x^2)^{\frac{n-1}{2}}dx$$

$$=V_{n-1}(1)\int_0^1 t^{\frac{n-1}{2}}(1-t)^{-\frac{1}{2}}dt=V_{n-1}(1)B\left(\frac{n+1}{2},\ \frac{1}{2}\right)$$

$$=V_{n-1}(1)\cdot\frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)}.$$

From this it follows that

$$V_n(1)=\frac{n^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)}.$$

924. The polynomials $1, x, \ldots, x^n$ form the basis. The square of the volume of the corresponding parallelepiped is

$$\begin{vmatrix} 1 & \frac{1}{2} & \cdots & \frac{1}{n+1} \\ \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n+2} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{1}{n+1} & \frac{1}{n+2} & \cdots & \frac{1}{2n+1} \end{vmatrix}=\frac{[1!\ 2!\ \ldots\ n!]^3}{(n+1)!\,(n+2)!\ \ldots\ (2n+1)!}.$$

925. (a) $\lambda_1=1$, $X_1=c\,(1,\ -1)$; $\lambda_2=3$, $X_2=c\,(1,\ 1)$;
(b) $\lambda_1=\ 7$, $X_1=c\,(1,\ 1)$; $\lambda_2=-2$, $X_2=c\,(4,\ -5)$;
(c) $\lambda_1=ai$, $X_1=c\,(1,\ i)$; $\lambda_2=-ai$, $X_2=c\,(1,\ -i)$;
(d) $\lambda_1=\ 2$, $X_1=c_1\,(1,\ 1,\ 0,\ 0)+c_2\,(1,\ 0,\ 1,\ 0)+c_3\,(1,\ 0,\ 0,\ 1)$;
$\lambda_2=-2$, $X_2=c\,(1,\ -1,\ -1,\ -1)$;
(e) $\lambda=2$, $X=c_1\,(-2,\ 1,\ 0)+c_2\,(1,\ 0,\ 1)$;
(f) $\lambda=-1$, $X=c\,(1,\ 1,\ -1)$;
(g) $\lambda_1=\ 1$, $X_1=c_1\,(1,\ 0,\ 1)+c_2\,(0,\ 1,\ 0)$; $\lambda_2=-1$, $X_2=c\,(1,\ 0,\ -1)$;
(h) $\lambda_1=0$, $X_1=c\,(3,\ -1,\ 2)$; $\lambda_{2,3}=\pm\sqrt{-14}$,
$X_{2,3}=c\,(3\pm2\sqrt{-14},\ 13,\ 2\mp3\sqrt{-14})$;
(i) $\lambda_1=1$, $X_1=c\,(3,\ -6,\ 20)$; $\lambda_2=-2$, $X_2=c\,(0,\ 0,\ 1)$;
(j) $\lambda_1=1$, $X_1=c\,(1,\ 1,\ 1)$; $\lambda_2=\varepsilon$, $X_2=c\,(3+2\varepsilon,\ 2+3\varepsilon,$
$3+3\varepsilon)$; $\lambda_3=\varepsilon^2$, $X_3=c\,(3+2\varepsilon^2,\ 2+3\varepsilon^2,\ 3+3\varepsilon^2)$

where

$$\varepsilon=-\frac{1}{2}+\frac{i\sqrt{3}}{2}.$$

926. The eigenvalues A^{-1} are reciprocals of the eigenvalues A. Indeed, from $|A^{-1}-\lambda E|=0$ it follows that $|E-\lambda A|=0$, $\left|A-\frac{1}{\lambda}E\right|=0$.

927. The eigenvalues of the matrix A^2 are equal to the squares of the eigenvalues for A. Indeed, let

$$|A-\lambda E|=(\lambda_1-\lambda)(\lambda_2-\lambda)\dots(\lambda_n-\lambda).$$

Then

$$|A+\lambda E|=(\lambda_1+\lambda)(\lambda_2+\lambda)\dots(\lambda_n+\lambda).$$

Multiplying these equations and substituting λ for λ^2, we get

$$|A^2-\lambda E|=(\lambda_1^2-\lambda)(\lambda_2^2-\lambda)\ \dots\ (\lambda_n^2-\lambda).$$

928. The eigenvalues of A^m are equal to the mth powers of the eigenvalues of A.

To see this, the simplest thing is to replace in the equation

$$|A-\lambda E|=(\lambda_1-\lambda)(\lambda_2-\lambda)\dots(\lambda_n-\lambda)$$

λ by $\lambda\varepsilon$, $\lambda\varepsilon^2$, ..., $\lambda\varepsilon^{n-1}$, where

$$\varepsilon=\cos\frac{2\pi}{n}+i\sin\frac{2\pi}{n},$$

multiply the equations and substitute λ for λ^n.

929. $f(A)=b_0(A-\xi_1E)\dots(A-\xi_mE)$, hence

$$|f(A)|=b_0^n\cdot|A-\xi_1E|\ \dots\ |A-\xi_mE|=b_0^nF(\xi_1)\ \dots\ F(\xi_m).$$

930. Let

$$F(\lambda)=|A-\lambda E|=(\lambda_1-\lambda)(\lambda_2-\lambda)\dots(\lambda_n-\lambda)$$

and

$$f(x)=b_0(x-\xi_1)(x-\xi_2)\dots(x-\xi_m).$$

Then

$$|f(A)|=b_0^n\prod_{i=1}^{n}\prod_{k=1}^{m}(\lambda_i-\xi_k)=f(\lambda_1)f(\lambda_2)\ \dots\ f(\lambda_n).$$

931. Put

$$\varphi(x)=f(x)-\lambda$$

and apply the result of the preceding problem.

We get

$$|f(A)-\lambda E|=(f(\lambda_1)-\lambda)(f(\lambda_2)-\lambda)\dots(f(\lambda_n)-\lambda)$$

whence it follows that the eigenvalues of the matrix $f(A)$ are $f(\lambda_1)$, $f(\lambda_2)$, ..., $f(\lambda_n)$.

932. Let X be an eigenvector of the matrix A corresponding to the eigenvalue λ.

Then

$$EX=X,$$
$$AX=\lambda X,$$
$$A^2X=\lambda^2X,$$
$$\dots\dots\dots$$
$$A^mX=\lambda^mX.$$

Multiplying these vector equations by arbitrary coefficients and combining, we get for any polynomial f

$$f(A)X=f(\lambda)\,X$$

i.e., X is the eigenvector of $f(A)$ corresponding to the eigenvalue $f(\lambda)$.

933. The eigenvalues of A^2 are n and $-n$ of multiplicities $\frac{n+1}{2}$ and $\frac{n-1}{2}$ respectively. Hence, the eigenvalues of A are $+\sqrt{n}$, $-\sqrt{n}$, $+\sqrt{ni}$ and $-\sqrt{ni}$. Let us denote their multiplicities by a, b, c, d. Then $a+b=\frac{n+1}{2}$, $c+d=\frac{n-1}{2}$. The sum of the eigenvalues of a matrix is equal to the sum of the elements of the principal diagonal.

Hence

$$[a-b+(c-d)\,i]\sqrt{n}=1+\varepsilon+\varepsilon^4+\ \dots\ +\varepsilon^{(n-1)^2}.$$

The modulus of the right side of this equation is equal to $\sqrt{n}$ (Problem 126). Consequently

$$(a-b)^2+(c-d)^2=1.$$

Since the numbers $c-d$ and $c+d$ are of the same parity, we conclude that

$$a-b=0,\ c-d=\pm 1 \text{ if } \frac{n-1}{2} \text{ is odd}$$

and

$$a-b=\pm 1,\ c-d=0 \text{ if } \frac{n-1}{2} \text{ is even.}$$

Hence, for $n=1+4k$

$$c=d=k,\quad a=k+1,\quad b=k \text{ or } a=k,\quad b=k+1$$

for $n=3+4k$

$$a=b=k+1,\quad c=k+1,\quad d=k \text{ or } c=k,\quad d=k+1.$$

Thus, the eigenvalues are determined to within sign. To determine the sign, take advantage of the fact that the product of the eigenvalues is equal to the determinant of the matrix. Using the result of Problem 299, it is easy to find that for $n=1+4k$

$$a=k+1,\quad b=k,$$

for $n=3+4k$

$$c=k+1,\quad d=k.$$

Thus, the eigenvalues are completely determined.

934. $1+\varepsilon+\varepsilon^4+\ \dots\ +\varepsilon^{(n-1)^2}=+\sqrt{n}$ for $n=4k+1$,

$$1+\varepsilon+\varepsilon^4+\ \dots\ +\varepsilon^{(n-1)^2}=+i\sqrt{n} \text{ for } n=4k+3.$$

935. (a) Put $\frac{x}{y}=\alpha^n$. Then

$$\lambda_k=y\,\frac{\alpha\varepsilon_k-\alpha^n}{1-\alpha\varepsilon_k}$$

where $\varepsilon_k=\cos\frac{2k\pi}{n}+i\sin\frac{2k\pi}{n}$, $k=0,\ 1,\ \dots,\ n-1$.

(b) $\lambda_k = a_1 + a_2\varepsilon_k + a_3\varepsilon_k^2 + \ \dots \ + a_n\varepsilon_k^{n-1}$.

(c) $\lambda_k = 2i \cos \frac{k\pi}{n+1}$, $k = 1, 2, \dots, n$.

936.

$$A \times B - \lambda E_{mn} = \begin{pmatrix} a_{11}B - \lambda E_m & a_{12}B & & a_{1n}B \\ a_{21}B & a_{22}B - \lambda E_m & \dots & a_{2n}B \\ \dots & \dots & \dots & \dots \\ a_{n1}B & a_{n2}B & \dots & a_{nn}B - \lambda E_m \end{pmatrix}$$

whence it follows, by the result of Problem 537, that

$$|A \times B - \lambda E_{mn}| = |\varphi(B)|$$

where

$$\varphi(x) = |Ax - \lambda E_n| = \prod_{i=1}^{n} (\alpha_i x - \lambda).$$

By the result of Problem 930,

$$|\varphi(B)| = \prod_{k=1}^{m} \varphi(\beta_k) = \prod_{i=1}^{n} \prod_{k=1}^{m} (\alpha_i\beta_k - \lambda).$$

Thus, the eigenvalues of $A \times B$ are the numbers $\alpha_i\beta_k$, where α_i are the eigenvalues of A, and β_k are the eigenvalues of B.

937. If A is a nonsingular matrix, then

$$|BA - \lambda E| = |A^{-1}(AB - \lambda E)A| = |A^{-1}| \cdot |AB - \lambda E| \cdot |A| = |AB - \lambda E|.$$

It is possible to get rid of the assumption that A is nonsingular by passing to the limit or by using the theorem on the identity of polynomials in many variables.

It is also possible, using the theorem on the multiplication of rectangular matrices, to compute directly the coefficients of the polynomials

$$|AB - \lambda E| \text{ and } |BA - \lambda E|$$

and satisfy yourself that they are equal.

938. Complete the matrices A and B to the square matrices A' and B' of order n by adjoining to A $n-m$ rows and to B $n-m$ columns made up of zeros. Then $BA = B'A'$ and $A'B'$ is obtained from AB by bordering with zeros. Using the result of the preceding problem yields what we set out to prove.

The solutions of the Problems 939, 940, 941 are not unique. The answers given below correspond to a transformation that least of all deviates from the "triangular".

939. (a) $x_1'^2 + x_2'^2 + x_3'^2$,

$x_1' = x_1 + x_2$,

$x_2' = x_2 + 2x_3$,

$x_3' = x_3$,

(b) $-x_1'^2 + x_2'^2 + x_3'^2$,

$x_1' = x_1$,

$x_2' = x_1 - 2x_2$,

$x_3' = x_1 + x_3$,

(c) $x_1'^2 - x_2^2 - x_3'^2$,

$$x_1' = \frac{1}{2}x_1 + \frac{1}{2}x_2 + x_3,$$
$$x_2' = \frac{1}{2}x_1 - \frac{1}{2}x_2,$$
$$x_3' = x_3,$$

(d) $x_1'^2 + x_2'^2 - x_3'^2$,

$$x_1' = x_1 - x_2 + x_3 - x_4,$$
$$x_2' = x_2 + x_3 + x_4,$$
$$x_3' = x_2 - x_3 + 2x_4,$$
$$x_4' = x_4,$$

(e) $x_1'^2 - x_2'^2 + x_3'^2 - x_4'^2$,

$$x_1' = x_1 + \frac{1}{2}x_2,$$
$$x_2' = \frac{1}{2}x_2,$$
$$x_3' = \frac{1}{2}x_3 + \frac{1}{2}x_4,$$
$$x_4' = \frac{1}{2}x_3 - \frac{1}{2}x_4.$$

940. $x_1'^2 + \frac{3}{4}x_2'^2 + \frac{4}{6}x_3'^2 + \ldots + \frac{n+1}{2n}x_n'^2.$

The variables $x_1', x_2', \ldots, x_n'$ are expressed linearly in terms of $x_1, x_2, \ldots, x_n$ with the matrix

$$\begin{Vmatrix} 1 & \frac{1}{2} & & \frac{1}{2} & \ldots & \frac{1}{2} \\ 0 & 1 & & \frac{1}{3} & \ldots & \frac{1}{3} \\ 0 & 0 & 1 & \frac{1}{4} & \ldots & \frac{1}{4} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \ldots & 1 \end{Vmatrix}.$$

941. $$\left(\frac{x_1+x_2}{2} + x_3 + x_4 + \ldots + x_n\right)^2 - \left(\frac{x_1 - x_2}{2}\right)^2 - \left(x_3 + \frac{1}{2}x_4 + \ldots + \frac{1}{2}x_n\right)^2 - \frac{3}{4}\left(x_4 + \frac{1}{3}x_5 + \ldots + \frac{1}{3}x_n\right)^2 - \ldots - \frac{n-1}{2(n-2)}x_n^2.$$

942. The matrix of a positive quadratic form is equal to $\bar{A}A$, where A is a nonsingular real matrix carrying the sum of squares to the given form. The positivity of the minors follows from the result of Problem 510.

943. Let
$$f=a_{11}x_1^2+\ \dots\ +a_{1n}x_1x_n$$
$$\dots\dots\dots\dots$$
$$+a_{n1}x_nx_1+\ \dots\ +a_{nn}x_n^2$$
be a quadratic form. We denote
$$f(k)=a_{11}x_1^2+\ \dots\ +a_{1k}x_1x_k$$
$$\dots\dots\dots\dots$$
$$+a_{k1}x_kx_1+\ \dots\ +a_{kk}x_k^2,$$

$$\Delta_k=\begin{vmatrix} a_{11} & \dots & a_{1k} \\ \cdot & \cdots & \cdot \\ a_{k1} & \dots & a_{kk}\end{vmatrix},$$ r is the rank of the form f.

Let
$$f=\alpha_1x_1'^2+\alpha_2x_2'^2+\ \dots\ +\alpha_nx_n'^2,$$
where
$$\begin{aligned} x_1 &= x_1+b_{12}x_2+\ \dots\ +b_{1n}x_n,\\ x_2' &= \qquad\quad x_2+\ \dots\ +b_{2n}x_n,\\ &\dots\dots\dots\dots\dots\\ x_n' &= \qquad\qquad\qquad\qquad x_n.\end{aligned}$$

Since the determinant of a triangular transformation is equal to 1, $D_f=\Delta_n=\alpha_1\alpha_2\dots\alpha_n$. Putting
$$x_{k+1}=\ \dots\ =x_n=0$$
we get
$$f^{(k)}=\alpha_1x_1^{(k)^2}+\alpha_2x_2^{(k)^2}+\ \dots\ +\alpha_kx_k^{(k)^2}$$
where
$$\begin{aligned} x_1^{(k)} &= x_1+b_{12}x_2+\ \dots\ +b_{1k}x_k,\\ x_2^{(k)} &= \qquad\quad x_2+\ \dots\ +b_{2k}x_k,\\ &\dots\dots\dots\dots\dots\\ x_k^{(k)} &= \qquad\qquad\qquad\qquad x_k.\end{aligned}$$

Whence it follows that $\Delta_k=\alpha_1\alpha_2\dots\alpha_k$ and that a necessary condition for the possibility of a triangular transformation to diagonal form is
$$\Delta_1\neq0,\ \Delta_2\neq0,\ \dots,\ \Delta_r\neq0.$$
It is easy to verify that this condition is sufficient.

Furthermore, $\alpha_k=\dfrac{\Delta_k}{\Delta_{k-1}}$ for $k\leqslant r$, $\alpha_k=0$ for $k>r$.

The discriminant of the form
$$f_k(x_{k+1},\ \dots,\ x_n)=f-\alpha_1x_1'^2-\ \dots\ -\alpha_kx_k'^2$$
$$=\alpha_{k+1}x_{k+1}'^2+\ \dots\ +\alpha_nx_n'^2$$
is equal to $\alpha_{k+1}\alpha_{k+2}\ \dots\ \alpha_n=\dfrac{\Delta_n}{\Delta_k}$.

944. The necessity of the Sylvester conditions was proved in Problem 942. The sufficiency follows from the result of Problem 943.

945. Let l be a linear form in the variables $x_1, x_2, \ldots, x_n$. Transform the form f by means of a transformation with determinant unity, taking the form l for the last of the new variables. Then perform a triangular transformation of the form f to canonical form.

The form f becomes

$$f=\alpha_1 x_1'^2+\alpha_2 x_2'^2+\ \ldots\ +\alpha_n x_n'^2$$

and $x_n'=l$.

The discriminant of the form f is equal to $\alpha_1\alpha_2\ldots\alpha_n$. The discriminant of the form $f+l^2$ is equal to $\alpha_1\alpha_2\ldots\alpha_{n-1}(\alpha_n+1)$. It is greater than the discriminant of the form f, since all the coefficients $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}, \alpha_n$ are positive.

946. $f(x_1, x_2, \ldots, x_n)$

$$=a_{11}x_1^2+2a_{21}x_1x_2+\ \ldots\ +2a_{n1}x_1x_n+\varphi$$

$$=a_{11}\left(x_1+\frac{a_{21}}{a_{11}}x_2+\ \ldots\ +\frac{a_{n1}}{a_{11}}x_n\right)^2+f_1(x_2, \ldots, x_n)$$

where

$$f_1=\varphi-a_{11}\left(\frac{a_{21}}{a_{11}}x_2+\ \ldots\ +\frac{a_{n1}}{a_{11}}x_n\right)^2.$$

The form f_1 is positive and its discriminant is equal to $\frac{D_f}{a_{11}}$ where D_f is the discriminant of f. On the basis of the result of Problem 945, $D_{f_1}\geqslant\frac{D_f}{a_{11}}$, which completes the proof.

947. The proof is the same as for the law of inertia.

948. Form the linear forms

$$l_k=u_1+u_2x_k+\ \ldots\ +u_nx_k^{n-1},\quad k=1, 2, \ldots, n$$

where $x_1, x_2, \ldots, x_n$ are roots of the given equation.

To equal roots will correspond equal forms, distinct roots are associated with linearly independent forms, real roots are associated with real forms, and conjugate complex roots are associated with conjugate complex forms.

The real and imaginary parts of the complex form $l_k=\lambda_k+\mu_k i$ will be linearly independent among themselves and also relative to all forms corresponding to roots distinct from x_k and x_k'.

Form the quadratic form

$$f(u_1, \ldots, u_n)=\sum_{k=1}^{n}(u_1+u_2x_i+\ \ldots\ +u_nx_i^{n-1})^2.$$

The rank of this form is equal to the number of distinct roots of the given equation. The matrix of its coefficients is

$$\begin{pmatrix} s_0 & s_1 & \ldots & s_{n-1} \\ s_1 & s_2 & \ldots & s_n \\ \ldots & \ldots & \ldots & \ldots \\ s_{n-1} & s_n & \ldots & s_{2n-2} \end{pmatrix}$$

The sum of squares of the conjugate complex linear forms $l_k=\lambda_k+$ $+i\mu_k$ and $l_k'=\lambda_k-i\mu_k$ is equal to $2\lambda_k^2-2\mu_k^2$. Hence, the number of negative squares in any (by the law of inertia) canonical representation of the form f

is equal to the number of distinct pairs of conjugate complex roots of the given equation.

949. This follows from the results of Problems 948, 944.

950. The operation (f, φ) is obviously distributive. It is therefore sufficient to carry out the proof for the squares of the linear forms.

Let

$$f=(\alpha_1x_1+\alpha_2x_2+\ \dots\ +\alpha_nx_n)^2,$$
$$\varphi=(\beta_1x_1+\beta_2x_2+\ \dots\ +\beta_nx_n)^2.$$

It is then easy to see that

$$(f\varphi)=(\alpha_1\beta_1x_1+\alpha_2\beta_2x_2+\ \dots\ +\alpha_n\beta_nx_n)^2\geqslant 0.$$

951. (a) $4x_1'^2+x_2'^2-2x_3'^2,\quad x_1'=\frac{2}{3}x_1-\frac{2}{3}x_2+\frac{1}{3}x_3,$

$$x_2'=\frac{2}{3}x_1+\frac{1}{3}x_2-\frac{2}{3}x_3,$$
$$x_3'=\frac{1}{3}x_1+\frac{2}{3}x_2+\frac{2}{3}x_3;$$

(b) $2x_1'^2-x_2'^2+5x_3'^2,\quad x_1'=\frac{2}{3}x_1-\frac{1}{3}x_2-\frac{2}{3}x_3,$

$$x_2'=\frac{2}{3}x_1+\frac{2}{3}x_2+\frac{1}{3}x_3,$$
$$x_3'=\frac{1}{3}x_1-\frac{2}{3}x_2+\frac{2}{3}x_3;$$

(c) $7x_1'^2+4x_2'^2+x_3'^2,\quad x_1'=\frac{1}{3}x_1+\frac{2}{3}x_2-\frac{2}{3}x_3,$

$$x_2'=\frac{2}{3}x_1+\frac{1}{3}x_2+\frac{2}{3}x_3,$$
$$x_3'=-\frac{2}{3}x_1+\frac{2}{3}x_2+\frac{1}{3}x_3;$$

(d) $10x_1'^2+x_2'^2+x_3'^2,\quad x_1'=\frac{1}{3}x_1+\frac{2}{3}x_2-\frac{2}{3}x_3,$

$$x_2'=\frac{2\sqrt{5}}{5}x_1-\frac{\sqrt{5}}{5}x_2,$$
$$x_3'=\frac{2\sqrt{5}}{15}x_1+\frac{4\sqrt{5}}{15}x_2+\frac{\sqrt{5}}{3}x_3;$$

(e) $-7x_1'^2+2x_2'^2+2x_3'^2,\quad x_1'=\frac{1}{3}x_1+\frac{2}{3}x_2-\frac{2}{3}x_3,$

$$x_2'=\frac{2\sqrt{5}}{5}x_1-\frac{\sqrt{5}}{5}x_2,$$
$$x_3'=\frac{2\sqrt{5}}{15}x_1+\frac{4\sqrt{5}}{15}x_2+\frac{\sqrt{5}}{3}x_3;$$

(f) $2x_1'^2+5x_2'^2+8x_3'^2,$

$$\begin{aligned} x_1' &= \frac{2}{3}x_1+\frac{1}{3}x_2+\frac{2}{3}x_3, \\ x_2' &= \frac{1}{3}x_1+\frac{2}{3}x_2-\frac{2}{3}x_3, \\ x_3' &= \frac{2}{3}x_1-\frac{2}{3}x_2-\frac{1}{3}x_3; \end{aligned}$$

(g) $7x_1'^2-2x_2'^2+7x_3'^2,$

$$\begin{aligned} x_1' &= \frac{\sqrt{2}}{2}x_1-\frac{\sqrt{2}}{2}x_3, \\ x_2' &= \frac{2}{3}x_1+\frac{1}{3}x_2+\frac{2}{3}x_3, \\ x_3' &= \frac{\sqrt{2}}{6}x_1-\frac{2\sqrt{2}}{3}x_2+\frac{\sqrt{2}}{6}x_3; \end{aligned}$$

(h) $11x_1'^2+5x_2'^2-x_3'^2,$

$$\begin{aligned} x_1' &= \frac{2}{3}x_1-\frac{2}{3}x_2-\frac{1}{3}x_3, \\ x_2' &= \frac{2}{3}x_1+\frac{1}{3}x_2+\frac{2}{3}x_3, \\ x_3' &= \frac{1}{3}x_1+\frac{2}{3}x_2-\frac{2}{3}x_3; \end{aligned}$$

(i) $x_1'^2-x_2'^2+3x_3'^2+5x_4'^2,$

$$\begin{aligned} x_1' &= \frac{1}{2}x_1+\frac{1}{2}x_2+\frac{1}{2}x_3+\frac{1}{2}x_4, \\ x_2' &= \frac{1}{2}x_1+\frac{1}{2}x_2-\frac{1}{2}x_3-\frac{1}{2}x_4, \\ x_3' &= \frac{1}{2}x_1-\frac{1}{2}x_2+\frac{1}{2}x_3-\frac{1}{2}x_4, \\ x_4 &= \frac{1}{2}x_1-\frac{1}{2}x_2-\frac{1}{2}x_3+\frac{1}{2}x_4; \end{aligned}$$

(j) $x_1'^2+x_2'^2-x_3'^2-x_4'^2,$

$$\begin{aligned} x_1' &= \frac{\sqrt{2}}{2}x_1+\frac{\sqrt{2}}{2}x_2, \\ x_2' &= \frac{\sqrt{2}}{2}x_3+\frac{\sqrt{2}}{2}x_4, \\ x_3' &= \frac{\sqrt{2}}{2}x_1-\frac{\sqrt{2}}{2}x_2, \\ x_4' &= \frac{\sqrt{2}}{2}x_3-\frac{\sqrt{2}}{2}x_4; \end{aligned}$$

(k) $x_1'^2+x_2'^2+3x_3'^2-x_4'^2,$

$$\begin{aligned}
x_1' &= \frac{\sqrt{2}}{2}x_2 + \frac{\sqrt{2}}{2}x_4,\\
x_2' &= \frac{\sqrt{2}}{2}x_1 + \frac{\sqrt{2}}{2}x_3,\\
x_3' &= \frac{1}{2}x_1 + \frac{1}{2}x_2 - \frac{1}{2}x_3 - \frac{1}{2}x_4,\\
x_4' &= -\frac{1}{2}x_1 + \frac{1}{2}x_2 + \frac{1}{2}x_3 - \frac{1}{2}x_4;
\end{aligned}$$

(l) $x_1'^2+x_2'^2+x_3'^2-3x_4'^2,$

$$\begin{aligned}
x_1' &= \frac{\sqrt{2}}{2}x_1 + \frac{\sqrt{2}}{2}x_2,\\
x_2' &= \frac{\sqrt{2}}{2}x_3 + \frac{\sqrt{2}}{2}x_4,\\
x_3' &= \frac{1}{2}x_1 - \frac{1}{2}x_2 + \frac{1}{2}x_3 - \frac{1}{2}x_4,\\
x_4' &= \frac{1}{2}x_1 - \frac{1}{2}x_2 - \frac{1}{2}x_3 + \frac{1}{2}x_4;
\end{aligned}$$

(m) $x_1'^2-x_2'^2+7x_3'^2-3x_4'^2,$

$$\begin{aligned}
x_1' &= \frac{1}{2}x_1 + \frac{1}{2}x_2 + \frac{1}{2}x_3 + \frac{1}{2}x_4,\\
x_2' &= \frac{1}{2}x_1 + \frac{1}{2}x_2 - \frac{1}{2}x_3 - \frac{1}{2}x_4,\\
x_3' &= \frac{1}{2}x_1 - \frac{1}{2}x_2 + \frac{1}{2}x_3 - \frac{1}{2}x_4,\\
x_4' &= \frac{1}{2}x_1 - \frac{1}{2}x_2 - \frac{1}{2}x_3 + \frac{1}{2}x_4;
\end{aligned}$$

(n) $5x_1'^2-5x_2'^2+3x_3'^2-3x_4'^2,$

$$\begin{aligned}
x_1' &= \frac{1}{2}x_1 + \frac{1}{2}x_2 + \frac{1}{2}x_3 + \frac{1}{2}x_4,\\
x_2' &= \frac{1}{2}x_1 + \frac{1}{2}x_2 - \frac{1}{2}x_3 - \frac{1}{2}x_4,\\
x_3' &= \frac{1}{2}x_1 - \frac{1}{2}x_2 + \frac{1}{2}x_3 - \frac{1}{2}x_4,\\
x_4' &= \frac{1}{2}x_1 - \frac{1}{2}x_2 - \frac{1}{2}x_3 + \frac{1}{2}x_1.
\end{aligned}$$

952. (a) $\frac{n+1}{2}x_1'^2+\frac{1}{2}(x_2'^2+x_3'^2+\ \dots\ +x_n'^2);$

(b) $\frac{n-1}{2}x_1'^2-\frac{1}{2}(x_2'^2+x_3'^2+\ \dots\ +x_n'^2)$

where

$$x_1' = \frac{1}{\sqrt{n}}(x_1+x_2+\ \dots\ +x_n);$$

$$x_i' = \alpha_{i1}x_1+\alpha_{i2}x_2+\ \dots\ +\alpha_{in}x_n,\quad i=2,\ \dots,\ n$$

where $(\alpha_{i1}, \alpha_{i2}, \dots, \alpha_{in})$ is any orthogonal and normalized fundamental system of solutions of the equation

$$x_1+x_2+\ \dots\ +x_n=0.$$

953. $x_1'^2\cos\frac{\pi}{n+1}+x_2'^2\cos\frac{2\pi}{n+1}+\ \dots\ +x_n'^2\cos\frac{n\pi}{n+1}\cdot$

954. If all the eigenvalues of the matrix A lie in the interval, $[a, b]$, then all the eigenvalues of the matrix $A-\lambda E$ are negative for $\lambda>b$ and positive for $\lambda<a$. Hence, a quadratic form with matrix $A-\lambda E$ is negative for $\lambda>b$ and positive for $\lambda<a$. Conversely, if the quadratic form $(A-\lambda E)X\cdot X$ is negative for $\lambda>b$ and positive for $\lambda<a$, then all the eigenvalues of the matrix $A-\lambda E$ are positive for $\lambda<a$ and negative for $\lambda>b$.

Consequently, all the eigenvalues of the matrix A lie in the interval $[a, b]$.

955. The following inequalities hold for any vector X:

$$aX\cdot X\leqslant AX\cdot X\leqslant cX\cdot X,\ bX\cdot X\leqslant BX\cdot X\leqslant dX\cdot X$$

whence $(a+b)\,X\cdot X\leqslant(A+B)\,X\cdot X\leqslant(c+d)\,X\cdot X$. Therefore, all the eigenvalues of the matrix $A+B$ lie in the interval $[a+c,\ b+d]$.

956. (a) This follows from the result of Problem 937.

(b) $|AX|^2=AX\cdot AX=X\ \cdot\bar{A}AX\leqslant|X|^2\cdot\|A\|^2$.

The equal sign occurs for the eigenvector of the matrix $\bar{A}A$ belonging to the eigenvalue $\|A\|^2$.

(c) $|(A+B)X|\leqslant|AX|+|BX|\leqslant(\|A\|+\|B\|)\,|X|$

for any vector X. But for some vector X_0,

$$|(A+B)\,X_0|=(\|A+B\|)\cdot|X_0|.$$

Consequently

$$\|A+B\|\leqslant\|A\|+\|B\|.$$

(d) $|ABX|\leqslant\|A\|\cdot|BX|\leqslant\|A\|\cdot\|B\|\cdot|X|$.

Applying this inequality to the vector X_0, for which

$$\|AB\|\cdot|X_0|=|ABX_0|$$

we get

$$\|AB\|\leqslant\|A\|\cdot\|B\|.$$

(e) Let $\lambda=p+qi$ be an eigenvalue of the matrix A, $X=Y+iZ$ the corresponding eigenvector.

Then

$$AY=pY-qZ,\ AZ=qY+pZ$$

whence

$$|pY-qZ|^2\leqslant\|A\|^2\ |Y|^2,$$

$$|qY+pZ|^2\leqslant\|A\|^2\ |Z|^2.$$

Combining these inequalities, we get

$$|\lambda|^2(|Y|^2+|Z|^2)=(p^2+q^2)(|Y|^2+|Z|^2)\leqslant\|A\|^2(|Y|^2+|Z|^2)$$

and, hence,

$$|\lambda| \leqslant \| A \|.$$

957. Let A be a real nonsingular matrix; then $\bar{A}AX \cdot X = |AX|^2$ is a positive quadratic form which can be brought to canonical form by a transformation of the variables with the triangular matrix B which has positive diagonal elements. Therefore, $\bar{A}A = \bar{B}B$, whence it follows that $\overline{AB^{-1}} \cdot AB^{-1} = E$, that is, $AB^{-1} = P$ is an orthogonal matrix. From this we have $A = PB$. Uniqueness follows from the uniqueness of the triangular transformation of a quadratic form to canonical form.

958. $\bar{A}A$ is a symmetric matrix with positive eigenvalues $\lambda_1, \ldots, \lambda_n$. Consequently,

$$\bar{A}A = P^{-1} \begin{pmatrix} \lambda_1 & & & & \\ & \lambda_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \lambda_n \end{pmatrix} P.$$

We construct the matrix

$$B = P^{-1} \begin{pmatrix} \mu_1 & & & & \\ & \mu_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \mu_n \end{pmatrix} P$$

where μ_i is the positive square root of λ_i. It is obvious that B is again a symmetric matrix with positive eigenvalues and $B^2 = \bar{A}A$. Whence it follows that $AB^{-1} = Q$ is an orthogonal matrix, $A = QB$.

959. After carrying the coordinate origin to the centre of the surface, the surface must contain point $-X$ along with point X, and, hence, the equation must not contain the running coordinates to first powers. After a parallel translation of the axes $X = X_0 + X'$, where X_0 is the translation vector, the equation of the surface becomes

$$AX' \cdot X' + 2(AX_0 + B)X' + AX_0 \cdot X_0 + 2BX_0 + C = 0.$$

Therefore, for the existence of a centre, it is necessary and sufficient that the equation $AX_0 + B$ be solvable with respect to the vector X_0, for which, in turn, it is necessary and sufficient that the rank of the matrix A be equal to the rank of the matrix (A, B).

960. After carrying the origin to the centre, the equation of the surface becomes

$$AX \cdot X + \gamma = 0.$$

If r is the rank of the matrix A and $\alpha_1, \ldots, \alpha_r$ are nonzero eigenvalues, then after an appropriate orthogonal transformation of the coordinates, the equation takes on the canonical form

$$\alpha_1 x_1^2 + \ldots + \alpha_r x_r^2 + \gamma = 0.$$

961. The surface has no centre if the rank of (A, B) is greater than the rank of A, which is only possible if $r = \text{rank } A < n$. Denote the whole space by R, the space AR by P and the orthogonal complement of P by Q. Then for any $Y \in Q$ we will have $AY = 0$ because

$$|AY|^2 = AY \cdot AY = Y \cdot AAY = 0$$

since $AAY \in P$. Let

$$B = B_1 + B_2, \quad B_1 \in P, \quad B_2 \in Q.$$

Then $B_2 \neq 0$, otherwise B would belong to P and the rank of (A, B) would be equal to r. Let $B_1 = AX_0$. After the translation $X = X_0 + X'$, the equation of the surface becomes

$$AX' \cdot X' + 2B_2X' + c' = 0.$$

Make one more translation $X' = aB_2 + X''$. Then

$$AX' \cdot X' = a^2AB_2 \cdot B_2 + 2aAB_2 \cdot X'' + AX'' \cdot X'' = AX'' \cdot X''$$

because $AB_2 = 0$ and, hence, the equation becomes

$$AX'' \cdot X'' + 2B_2X'' + 2a\,|B_2|^2 + c' = 0.$$

Take

$$a = -\frac{c'}{2\,|B_2|^2}.$$

Then the equation becomes

$$AX'' \cdot X'' + 2B_2X'' = 0.$$

Now perform an orthogonal transformation of the coordinates, taking pairwise orthogonal unit eigenvectors of the matrix A for the basis of the space, including in them the unit vector that is collinear with the vector B_2 and in the opposite direction. This can be done because B_2 is an eigenvector of A. On this basis, the equation becomes

$$\lambda_1x_1^2 + \ldots + \lambda_rx_r^2 - 2\beta_2x_{r+1} = 0$$

where $\beta_2 = |B_2|$. It remains to divide by β_2.

962. Under a linear transformation with the matrix A, the space is mapped onto the subspace spanned by the vectors $A_1, A_2, \ldots, A_n$ whose coordinates form the columns of A. The required result follows immediately.

963. Let $e_1, e_2, \ldots, e_q$ be the basis of Q. Then Q' is a space spanned by $e_1', e_2', \ldots, e_q'$, where $e_1', e_2', \ldots, e_q'$ are the images of $e_1, e_2, \ldots, e_q$ under the linear transformation thus performed. Hence, $q' \leqslant q$. Besides, it is obvious that $q' \leqslant r$ because Q' lies in R'. Furthermore, let P' be the complementary space of Q, of dimension $p = n - q$, and let p' be its image under a linear transformation. Its dimension p' does not exceed $n - q$. But $P' + Q' = R'$, hence, $p' + q' \geqslant r$. From this,

$$q' \geqslant r - p' \geqslant r + q - n.$$

964. Let the rank of A be r_1, the rank of B be r_2 and let $BR = Q$. The dimensionality of Q is equal to r_2. Then ρ, which is equal to the rank of AB, is the dimensionality of $ABR = AQ$. By virtue of the result of the preceding problem, $r_1 + r_2 - n \leqslant \rho \leqslant \min(r_1, r_2)$.

965. A double performance of the operation of projection is equivalent to a single performance. Indeed, in the first projection, all the vectors of the space R go into the vectors of the subspace P, which under the second projection remain fixed. Hence, $A^2 = A$. Conversely, let $A^2 = A$. Denote by P the set of all vectors $Y = AX$, by Q the set of vectors Z such that $AZ = 0$. It is obvious that P and Q are linear spaces. Their intersection is the zero vector, because if $AX = Z$, then $AX = A^2X = AZ = 0$. Furthermore, for any vector X we have the expansion $X = AX + (E - A)X$. It is obvious that $(E - A)X \in Q$ because $A(E - A)X = (A - A^2)X = 0$. Therefore, $P + Q$ is the whole space, that is, P and Q are mutually complementary subspaces. The operation AX is a transition from the vector X to its component in P, that is, it is the operation of projection on P parallel to Q.

966. Let $P \perp Q$. Choose an orthonormal basis of the whole space by combining the orthonormal bases of P and Q. In this basis, the projection matrix will have the diagonal form

$$A' = \begin{Vmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & \ddots & & & & & \\ & & & 1 & & & & \\ & & & & 0 & & & \\ & & & & & \ddots & & \\ & & & & & & 0 \end{Vmatrix}.$$

In any other orthonormal basis, the projection matrix is equal to $A = B^{-1}A'B$, where B is some orthogonal matrix. Obviously, A is symmetric. Conversely, if A is a symmetric matrix and $A^2 = A$, then the spaces $P = AR$ and $Q = (E - A) R$ are orthogonal, because

$$AX(E-A)\,Y = X \cdot \tilde{A}\,(E-A)\,Y = X\,(A-A^2)\,Y = 0.$$

967. Let A be a skew-symmetric matrix. It is easy to see that $AX \cdot X = 0$ for any real vector X, because

$$AX \cdot X = X \cdot \tilde{A}X = X \cdot (-AX) = -AX \cdot X.$$

Let $\lambda = \alpha + \beta i$ be an eigenvalue of the matrix A and $U = X + Yi$ its corresponding eigenvector. Then

$$AX = \alpha X - \beta Y,\ AY = \beta X + \alpha Y.$$

From this it follows that $\alpha(|X|^2 + |Y|^2) = AX \cdot X + AY \cdot Y = 0$ and $\alpha = 0$. Furthermore, $\beta X \cdot Y + \alpha |Y|^2 = AY \cdot Y = 0$, whence $X \cdot Y = 0$ for $\beta \neq 0$. Finally, from the equality $\beta(|X|^2 - |Y|^2) = AY \cdot X + AX \cdot Y = AY \cdot X - X \cdot AY = 0$ follows $|X| = |Y|$.

968. Let

$$A = \begin{pmatrix} 0 & a_{12} & \dots & a_{1n} \\ -a_{12} & 0 & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ -a_{1n} & -a_{2n} & \dots & 0 \end{pmatrix}$$

be a skew-symmetric matrix. If all its eigenvalues are equal to zero, then $A = 0$. Indeed, the sum of the products of all eigenvalues taken two at a time is equal to the sum of all principal minors of the second order $\sum\limits_{i<k} a_{ik}^2$ and the fact that this sum is zero implies $a_{ik} = 0$ for any i, k, that is, $A = 0$.

Let A have a nonzero eigenvalue $\lambda_1 = a_1 i$. Normalize the real and imaginary parts of the eigenvector belonging to it. Because of the equality of their lengths, the normalizing factor will be the same, and the equations

$$AX = -a_1 Y,\ AY = a_1 X$$

will hold for the resulting vectors.

Form an orthogonal matrix by putting the vectors X and Y in the first two columns. Then

$$P^{-1}AP=\begin{pmatrix} 0 & a_1 & \dots \\ -a_1 & 0 & \dots \\ 0 & 0 & \dots \\ \dots & \dots & \dots \\ 0 & 0 & \dots \end{pmatrix}.$$

Since the matrix $P^{-1}AP$ is skew-symmetric, all nonindicated elements of the first two rows are zero, and the matrix of the elements of the 3rd, 4th, ..., nth row and the 3rd, 4th, ..., nth column will be skew-symmetric. Arguing in similar fashion, we isolate yet another submatrix $\begin{pmatrix} 0 & a_2 \\ -a_2 & 0 \end{pmatrix}$. The process is continued until the lower left corner has a matrix all eigenvalues of which are zero. But all the elements of such a matrix are zero. The problem is solved.

969. Let

$$B=(E-A)(E+A)^{-1}.$$

Then

$$\bar{B}=(\overline{E+A})^{-1}(\overline{E-A})=(E-A)^{-1}(E+A)=B^{-1}.$$

Furthermore

$$B+E=(E-A)(E+A)^{-1}+(E+A)(E+A)^{-1}=2(E+A)^{-1}$$

and, hence,

$$|B+E|\neq 0.$$

Conversely, if

$$\bar{B}=B^{-1} \text{ and } |B+E|\neq 0$$

then for A we can take $(E+B)^{-1}(E-B)$. It is easy to see that A is a skew-symmetric matrix.

970. Let A be an orthogonal matrix. Then

$$AX\cdot AY=X\cdot\bar{A}AY=X\cdot Y$$

for any real vectors X and Y. Let

$$\lambda=\alpha+\beta i$$

be an eigenvalue of the matrix A, and

$$U=X+Yi$$

a corresponding eigenvector. Then

$$AX=\alpha X-\beta Y,\quad AY=\alpha Y+\beta X$$

whence

$$|X|^2=X\cdot X=AX\cdot AX=\alpha^2|X|^2+\beta^2|Y|^2-2\alpha\beta X\cdot Y,$$

$$|Y|^2=Y\cdot Y=AY\cdot AY=\alpha^2|Y|^2+\beta^2|X|^2+2\alpha\beta X\cdot Y.$$

Adding these equations, we get $\alpha^2+\beta^2=1$.

971. For $\beta\neq 0$, we get, from the last equations of the preceding problem,

$$\beta(|X|^2-|Y|^2)+2\alpha X\cdot Y=0.$$

On the other hand,

$$X\cdot Y=AX\cdot AY=(\alpha^2-\beta^2)X\cdot Y+\alpha\beta(|X|^2-|Y|^2)$$

whence

$$\alpha(|X|^2-|Y|^2)-2\beta X\cdot Y=0.$$

Therefore

$$X\cdot Y=|X|^2-|Y|^2=0.$$

972. 1. Let $\lambda=\alpha+\beta i=\cos\varphi+i\sin\varphi$ be a complex eigenvalue of the matrix. Form an orthogonal matrix A, the first two columns of which constitute the real and imaginary parts of an eigenvector belonging to λ. Then

$$Q^{-1}AQ=\begin{pmatrix} \cos\varphi & \sin\varphi & \dots \\ -\sin\varphi & \cos\varphi & \dots \\ 0 & 0 & \dots \\ 0 & 0 & \dots \\ \dots & \dots & \dots \\ 0 & 0 & \dots \end{pmatrix}.$$

Because of the orthogonality of the matrix $Q^{-1}AQ$, the sum of the squares of the elements of each row is equal to 1, and, hence, all nonindicated elements of the first two rows are equal to 0.

2. Let $\lambda=\pm 1$ be a real eigenvalue of the matrix A, and let X be a normalized eigenvector belonging to λ.

Form an orthogonal matrix, the vector X constituting the first column. Then

$$Q^{-1}AQ=\begin{pmatrix} \pm 1 & \dots \\ 0 & \dots \\ \dots & \dots \\ 0 & \dots \end{pmatrix}.$$

All nonindicated elements of the first row are zero since the sum of the squares of the elements of each row of the orthogonal matrix $Q^{-1}AQ$ is equal to unity.

Applying the foregoing reasoning to the orthogonal matrices which remain in the lower left-hand corner, we get the required result.

973.

(a) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, (b) $\begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, (c) $\begin{pmatrix} -3 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$,

(d) $\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$, (e) $\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$, (f) $\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$,

(g) $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, (h) $\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, (i) $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$,

(j) $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, (k) $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, (l) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix}$,

(m) $\begin{pmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, (n) $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, (o) $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

974.

(a) $\begin{pmatrix} 1 & 1 & & \\ 0 & 1 & & \\ & 1 & 1 & \\ & 0 & 1 & \end{pmatrix}$, (b) $\begin{pmatrix} 1 & 1 & & \\ 0 & 1 & 1 & \\ & 0 & 1 & 1 \\ & & 0 & 1 \end{pmatrix}$, (c) $\begin{pmatrix} \varepsilon & & & & \\ & \varepsilon^2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \varepsilon^{n-1} \end{pmatrix}$

where $\varepsilon = \cos \dfrac{2\pi}{n} + i \sin \dfrac{2\pi}{n}$.

975. The submatrix $\begin{pmatrix} \lambda_i & 1 & & \\ & \lambda_i & 1 & \\ & & \ddots & \ddots \\ & & & \cdot\ 1 \\ & & & \lambda_i \end{pmatrix}$ cannot be periodic.

Remark. The result is not valid in a field with nonzero characteristic. For example, in a field of characteristic 2 we have

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^2 = E.$$

976. Let A be a given matrix and let $B = C^{-1}AC$ be its reduction to Jordan canonical form. The canonical matrix B is of triangular form and its diagonal elements are equal to the eigenvalues of the matrix A, each one being repeated as many times as its multiplicity in the characteristic equation. Furthermore, $B'_m = (C'_m)^{-1}A'_mC'_m$ (Problem 531). Consequently, the characteristic polynomials of the matrices A'_m and B'_m coincide. Given an appropriate numbering of combinations, the matrix B'_m has triangular form and, hence, its eigenvalues are equal to the diagonal elements. They are also, evidently, equal to all possible products of the eigenvalues of the matrix A taken m at a time.

977. The matrices $A - \lambda E$ and $\bar{A} - \lambda E$ obviously have coincident elementary divisors. Therefore, A and $\tilde{A}$ are convertible to one and the same canonical matrix and, hence, they are interconvertible.

978. If $A = CD$, where C, D are symmetric matrices and the matrix C is nonsingular, then $\bar{A} = DC$ and, hence, $\bar{A} = C^{-1}AC$. Thus, the matrix C should be sought among matrices that transform A into $\bar{A}$.

Let $A = SBS^{-1}$, where B is a canonical matrix:

$$B = \begin{pmatrix} B_1 & & & & \\ & B_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & B_k \end{pmatrix} \quad \text{where } B_i = \begin{pmatrix} \lambda_i & 1 & & \\ & \lambda_i & 1 & \\ & & \ddots & \ddots \\ & & & \cdot \\ & & & \lambda_i^1 \end{pmatrix}.$$

Then $\bar{A}=\bar{S}^{-1}\bar{B}\bar{S}$. Denote by H_i the matrix

$$\begin{pmatrix} & & & & 1 \\ & & & 1 & \\ & & \cdot & & \\ & \cdot & & & \\ 1 & & & & \end{pmatrix}.$$

It is easy to see that $\bar{B}_i=H_i^{-1}B_iH_i$ and, therefore,

$$\bar{B}=H^{-1}BH \text{ where } H=\begin{pmatrix} H_1 & & & \\ & H_2 & & \\ & & \ddots & \\ & & & H_k \end{pmatrix}.$$

Thus, $\bar{A}=\bar{S}^{-1}\ H^{-1}\ BH\bar{S}=\bar{S}^{-1}H^{-1}S^{-1}\ ASH\bar{S}=C^{-1}AC$ where $C=SH\bar{S}$. The matrix C is obviously symmetric. Put $D=C^{-1}A$. Then $\bar{D}=\bar{A}C^{-1}=$ $=C^{-1}ACC^{-1}=D$. Thus, the matrix D is also symmetric and $A=CD$.

979. Let $|A-\lambda E|=(-1)^n(\lambda^n-c_1\lambda^{n-1}-c_2\lambda^{n-2}-\ldots-c_n)$. It is obvious that $p_1=\operatorname{tr} A=c_1$. Suppose we have already proved that $p_1=$ $=c_1$, $p_2=c_2$, ..., $p_{k-1}=c_{k-1}$. Under this assumption, we shall prove that $p_k=c_k$. By construction, $A_k=A^k-p_1A^{k-1}-p_2A^{k-2}-\ldots-p_{k-1}A=A^k-$ $-c_1A^{k-1}-c_2A^{k-2}-\ldots-c_{k-1}A$. Hence

$$\operatorname{tr} A_k=kp_k=\operatorname{tr} A^k-c_1\operatorname{tr} A^{k-1}-\ldots-c_{k-1}\operatorname{tr} A$$
$$=S_k-c_1S_{k-1}-\ldots-c_{k-1}S_1$$

where $S_1, S_2, \ldots, S_k$ are power sums of the eigenvalues of the matrix A. But by Newton's formulas, $S_k-c_1S_{k-1}-\ldots-c_{k-1}S_1=kc_k$. Consequently,

$$p_k=c_k.$$

Next, $B_n=A^n-c_1A^{n-1}-\ldots-c_{n-1}A-c_nE=0$ by the Hamilton-Cayley relation. Finally,

$$AB_{n-1}=A_n=c_nE$$

whence

$$B_{n-1}=c_nA^{-1}.$$

980. Let

$$C=\begin{pmatrix} 0 & c_{12} & \ldots & c_{1n} \\ c_{21} & 0 & \ldots & c_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ c_{n1} & c_{n2} & \ldots & 0 \end{pmatrix}.$$

We consider the diagonal matrix

$$X=\begin{pmatrix} \alpha_1 & & & 0 \\ & \alpha_2 & & \\ & & \ddots & \\ 0 & & & \alpha_n \end{pmatrix}$$

the diagonal elements of which are arbitrary but pairwise distinct. Let us take

$$Y=\begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \dots & \dots & \dots & \dots \\ b_{n1} & b_{n2} & \dots & b_{nn} \end{pmatrix}.$$

Then

$$XY-YX=\begin{pmatrix} 0 & b_{12}(\alpha_2-\alpha_1) & \dots & b_{1n}(\alpha_n-\alpha_1) \\ b_{21}(\alpha_1-\alpha_2) & 0 & \dots & b_{2n}(\alpha_n-\alpha_2) \\ \dots & \dots & \dots & \dots \\ b_{n1}(\alpha_1-\alpha_n) & b_{n2}(\alpha_2-\alpha_n) & \dots & 0 \end{pmatrix}$$

and, consequently, it suffices to take $b_{ik}=\frac{c_{ik}}{\alpha_k-\alpha_i}$ for $i\neq k$. Now, using mathematical induction, we establish that any matrix with trace zero is similar to a matrix all diagonal elements of which are zero. Since $\operatorname{tr} C=0$, $C\neq \mu E$ for $\mu\neq 0$, and, hence, there is a vector U such that CU and U are linearly independent. Including the vectors U and CU in the basis, we find that C is similar to the matrix

$$C'=\begin{pmatrix} 0 & \gamma_{12} & \gamma_{13} & \dots & \gamma_{1n} \\ 1 & \gamma_{22} & \gamma_{23} & \dots & \gamma_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \gamma_{n2} & \gamma_{n3} & \dots & \gamma_{nn} \end{pmatrix}=\begin{pmatrix} 0 & P \\ Q & \Gamma \end{pmatrix}.$$

It is obvious that $\operatorname{tr}\Gamma=\operatorname{tr} C=0$.

Hence, by the induction hypothesis, $\Gamma=S^{-1}\Gamma' S$ where Γ' is a matrix with zero diagonal elements. Then all the diagonal elements are zero in the matrix

$$C''=\begin{pmatrix} 1 & 0 \\ 0 & S \end{pmatrix}^{-1} C'\begin{pmatrix} 1 & 0 \\ 0 & S \end{pmatrix}=\begin{pmatrix} 0 & PS \\ S^{-1}Q & S^{-1}\Gamma S \end{pmatrix}.$$

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